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Discrete Optimization

Dynamic programming and minimum risk paths

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Abstract

This paper addresses the problem of computing minimum risk paths by taking as objective the expected accident cost. The computation is based on a dynamic programming formulation which can be considered an extension of usual dynamic programming models: path costs are recursively computed via functions which are assumed to be monotonic. A large part of the paper is devoted to analyze in detail this formulation and provide some new results. Based on the dynamic programming model a linear programming model is also presented to compute minimum risk paths. This formulation turns out to be useful in solving a biobjective version of the problem, in which also expected travel length is taken into consideration. This leads to define nondominated mixed strategies. Finally it is shown how to extend the basic updating device of dynamic programming in order to enumerate all nondominated paths.

Keywords: Routing; Risk analysis; Dynamic programming; Multiple objective programming; Hazardous materials

1. Introduction

The travelling of hazardous materials has raised the problem of computing paths minimizing not only the length (cost or time) but also the risk of damages caused by accidents. Defining the length of a path and computing a minimal path length is clearly a standard issue. On the contrary there are various ways of defining and computing the "risk" of a path. The paper by Erkut and Verter (1998) reviews in detail the various approaches suggested in the literature and tests them on a real example.

We summarize the relevant facts. Two quantities are typically involved in assessing the risk: the probability of accident occurrence on a certain path edge and the incurred cost in case of accident on that path edge. Although it is not straightforward how to measure the cost, we assume that numbers c_e, related to the accident costs, are available for each path edge e.

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Two simple ways of taking care of the risk consider just one of the two quantities. For instance we may consider that accident costs are high on any path edge and consequently we want to minimize the probability of an accident along a path P. If p_e is the probability of an accident on the edge e, we have to compute minimal length paths with 'length' given by $\sum_{e \in P} - \log(1 - p_e)$. Alternatively we may consider that sooner or later an accident will occur by repeated travelling and therefore we may find more appropriate to minimize either $\sum_{e \in P} c_e$ or $\max_{e \in P} c_e$. The two quantities can also be considered together as a bicriterion problem (i.e. considering efficient solutions of minimizing at the same time both the accident probability and the cost).

Furthermore, accident probabilities and costs can be combined to evaluate the expected cost of an accident. This is the so called path risk (Erkut and Verter (1998)). The expected cost on path $P = \{e_1, e_2, \ldots, e_m\}$ is defined as

$$c(P) := p_1 c_1 + (1 - p_1) p_2 c_2 + (1 - p_1)(1 - p_2) p_3 c_3 + (1 - p_1)(1 - p_2)(1 - p_3) p_4 c_4 + \cdots, \tag{1}$$

where it is explicitly taken into account the fact that once an accident has occurred on path edge e_k no further transportation will take place on the subsequent path edges e_h, $h > k$.

This paper has been first motivated by the attempt to compute paths according to this expression without simplifying it. Erkut and Verter (1998) disregard higher order terms in (1) thus arriving to the following simple linear expression for the path expected cost: $\sum_{e \in P} p_e c_e$. They find this approximation both realistic given the small probability values in real life problems and also convenient because (1) is otherwise computed by resorting to a complex nonlinear integer programming problem or to a larger linear integer programming problem.

We want to show in this paper that dynamic programming can be easily used to compute minimal paths according to (1) in polynomial time. Although it can be practical to simplify such a cumbersome expression, we think that it is worthwhile facing the problem directly. The first observation is that (1) can be more compactly written recursively (in backward form)

$$c(P) := p_1 c_1 + (1 - p_1) c(P \setminus e_1). \tag{2}$$

This particular definition of path cost is particularly suited to dynamic programming. However, the dynamic programming formulation we need is slightly more general than the one found in the literature and we need to address in detail this issue in Section 2. As we shall see, there are some subtle points connected to this formulation. There are some strong connections between the path cost definition introduced in this paper and the generalized path algebra introduced by Gondran and Minoux (1979). In this paper we investigate the link between dynamic programming principles and techniques and a general path cost definition and provide new results.

In Section 3 we apply the dynamic programing techniques to solve (2) and discuss the solution. There are similarities between (2) and Markov decision processes (for a general reference to MDP see for instance Puterman (1994)). The difference is that in MDP paths are random, i.e. they are not known in advance, whereas here paths are deterministic (except for the possible interruption caused by an accident). We will show also a linear programming formulation of the problem.

With respect to the problem of finding the 'best' path for the travelling of hazardous materials it is certainly more interesting to address the problem as a bicriterion one. In Section 4 we consider bicriterion minimal paths and introduce the idea, borrowed from MDP and game theory, of using mixed strategies, that is using alternative paths with random selection (for the concept of mixed strategies refer for instance to Luce and Raiffa (1957)). The linear programming formulation turns out to be naturally suited to this approach. If mixed strategies cannot be considered we present a biobjective version of the basic updating mechanism of dynamic programming algorithms.

2. Extending the scope of dynamic programming

Let $G = (N, E)$ be a directed graph (let $n = |N|$ and $m = |E|$) and s a distinguished node of G. We consider directed paths starting from s. This corresponds to a forward dynamic programming model. In case of a backward model we have a