Computers & Operations Research 36 (2009) 2418 – 2428

Contents lists available at ScienceDirect

Computers & Operations Research

journal homepage: www.elsevier.com/locate/cor

On stochastic dynamic programming for solving large-scale planning problems under uncertainty☆

M. Pilar Cristobal[a], Laureano F. Escudero[b,*], Juan F. Monge[c]

[a]Dpto. de Matemtica Aplicada, ETSI de Montes, Universidad Politcnica de Madrid, Ciudad Universitaria, 28040 Madrid, Spain
[b]Dpto. De Estadística e Investigacíon-Operativa, Universidad Rey Juan Carlos, c/Tulipan, 28933 Móstoles (Madrid), Spain
[c]Centro de Investigación Operativa, Universidad Miguel Hernández, Ave de la Universidad, 03203 Elche (Alicante), Spain

ARTICLE INFO

Available online 6 October 2008

Keywords:
Stochastic dynamic programming
Scenario tree
Mixed 0–1 model
Tactical production planning

ABSTRACT

The stochastic dynamic programming approach outlined here, makes use of the scenario tree in a back-to-front scheme. The multi-period stochastic problems, related to the subtrees whose root nodes are the starting nodes (i.e., scenario groups), are solved at each given stage along the time horizon. Each subproblem considers the effect of the stochasticity of the uncertain parameters from the periods of the given stage, by using curves that estimate the expected future value (*EFV*) of the objective function. Each subproblem is solved for a set of reference levels of the variables that also have nonzero elements in any of the previous stages besides the given stage. An appropriate sensitivity analysis of the objective function for each reference level of the linking variables allows us to estimate the *EFV* curves applicable to the scenario groups from the previous stages, until the curves for the first stage have been computed. An application of the scheme to the problem of production planning with logical constraints is presented. The aim of the problem consists of obtaining the planning of tactical production over the scenarios along the time horizon. The expected total cost is minimized to satisfy the product demand. Some computational experience is reported. The proposed approach compares favorably with a state-of-the-art optimization engine in instances on a very large scale.

Scope and purpose

For quite some time, we have known that traditional methods of deterministic optimization are not suitable to capture the truly dynamic nature of most real-life problems, in view of the fact that the parameters which represent information concerning the future are uncertain. Many of the problems in planning under uncertainty, have logical constraints that require 0–1 variables in their formulation and can be solved via stochastic integer programming using scenario tree analysis. Given the dimensions of the deterministic equivalent model (*DEM*) of the stochastic problem, certain decomposition approaches can be considered by exploiting the structure of the models. Traditional decomposition schemes, such as the Benders and Lagrangean approaches, do not appear to provide the solution for large-scale problems (mainly in the cardinality of the scenario tree) in affordable computing effort. In this work, a stochastic dynamic programming approach is suggested, which we feel is particularly suited to exploit the scenario tree structure and, therefore, very amenable to finding solutions to very large-scale *DEMs*. The pilot case used involves a classical tactical production planning problem, where the structure is not exploited by the proposed approach so that it is generally applicable.

1. Introduction

Let the following dynamic multi-linking constraint deterministic program,

$$
\begin{aligned}
\min \quad & \sum_{t\in\mathcal{T}} c_t x_t \\
\text{s.t.} \quad & \sum_{t\in\mathcal{T}:t\leqslant\tau} A^t_\tau x_t = b_\tau \quad \forall\tau\in\mathcal{T}, \\
& x_t\in\mathcal{X} \quad \forall t\in\mathcal{T},
\end{aligned}
\qquad (1.1)
$$

☆ This research has been partially supported by the projects GRUPOS79/04 from the Generalitat Valenciana and URJC-CM-2007-CT-1622 from the Comunidad de Madrid, Spain.

* Corresponding author. Tel.: +34 9166 47457; fax: +34 914 88 7626.
E-mail addresses: pilar.cristobal@upm.es (M.P. Cristobal), Laureano.escudero@urjc.es (L.F. Escudero), monge@umh.es (J.F. Monge).

0305-0548/$ - see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.cor.2008.09.009

where $\mathscr{T}$ is the set of periods in a given time horizon, x_t is the vector of the variables related to time period t, c_t is the row vector of the objective function coefficients, A^t_τ is the constraint matrix related to time period τ for the variables x_t, $\mathscr{X}$ is the set of feasible solutions including the nonnegativity and the 0–1 values for the variables, and b_τ is the right-hand-side (*rhs*) vector for the constraints related to time period τ, for $\tau \in \mathscr{T}$, all with the appropriate dimensions. A particular case for variables with nonzero elements in constraints related to two time periods, not necessarily consecutive, is presented in [1] for the case of multi-level product supplying with a transportation time interval greater than two periods. A typical case with variables having nonzero elements in constraints related to two consecutive time periods is when goods are stocked in one period to be used in the following, see [2,3], among many others. See also Section 5.

However, very frequently some of the parameters (mainly, the objective function coefficients and the *rhs*) are not known with certainty when the decision is to be taken. With the aid of today's state-of-the-art optimization tools, deterministic mixed integer programs (MIP) should not present major difficulties for solving moderate size cases, at least. Moreover, it has been accepted for a long time that traditional deterministic optimization is not suitable for capturing the truly dynamic nature of most real-life applications. The main reason being that such applications involve, as previously stated, data uncertainties which arise because information that will occur in subsequent decision stages is not available to the decision maker when the decision must be made. For our purposes it suffices to consider the uncertainty of the vectors b and c. The stochastic problem will be dealt with using a scheme, such that the uncertainty of the parameters is visualized by a scenario tree.

Stochastic integer programming has a broad field of application and is flourishing in such sectors as finance, telecommunications, hydrothermal electricity generation, oil, hydrocarbon and chemical supplying, transformation and distribution logistics, revenue management, strategic and tactical production planning, supply chain management, and site and road location in addition to other sectors. See particularly the books [4–7], among others. Given the dimensions of the cases, certain decomposition approaches are considered, most of them based on Benders, Lagrangean and branch-and-fix coordination decomposition schemes for the structured mixed 0–1 *deterministic equivalent model* (*DEM*); for recommended text books on stochastic programming, see [8–10], among others.

Moreover, decomposition schemes based on the above approaches do not seem to provide the solution for large-scale problems (mainly, in the scenario tree) in reasonable computing effort. Alternatively, some kinds of stochastic dynamic programming (*SDP*) have been used for solving water resource management problems, see in a different context [11–15]. See in [16,17] the work which provided the inspiration for this paper.

The purpose of this paper is to present an *SDP* approach that utilizes a scenario tree *back-to-front* scheme. It obtains the solution of the multi-period stochastic mixed 0–1 subproblems related to the subtrees whose root nodes are the starting nodes (i.e., scenario groups) in each stage along the time horizon. Each subproblem considers the effect of the stochasticity of the uncertain parameters from the given stage, by using curves that give the *expected future value* (*EFV*) of the objective function. Each subproblem is solved for a representative set of *reference levels* of the linking variables between the previous stages and a given one. An appropriate sensitivity analysis of the objective function for a set of *reference levels* of the variables allows us to estimate the *EFV* curves for the scenario groups from the previous stages, until the curves for the first stage are computed. In this way, the *EFV* curves of the variables for the implementable time periods (i.e., stage 1) are obtained by considering all scenarios, but without being subordinated to any of them. The solution to be obtained for the first stage considers the influence of the scenarios by using the *EFV* curves that have been obtained. So, the original stochastic mixed 0–1 problem is broken down into as many subproblems as subtrees that there are in the scenario tree, where the roots are the starting nodes of each stage and the successor nodes belong to that stage. The 0–1 variables and the continuous variables are allowed for at any stage in the time horizon. The scope of this paper only considers the continuous linking variables.

Additionally, an application of the proposed methodology is considered for the classical problem of tactical production planning. This application will be used as a pilot case to assess the validity of the scheme proposed in the paper. The goal consists of determining the production and stocking of a set of products to satisfy an uncertain demand along a time horizon, at a minimum expected cost, subject to resource availability and logical constraints, among others. The uncertain parameters are the product cost and demand and the resource available along the time horizon. The uncertainty is represented by a scenario tree. The aggregate version of the *DEM* is presented as a multi-stage mixed 0–1 program, where the 0–1 variables are the tactical variables (decisions on products and lot sizing), and the continuous variables represent the production, stock and lost demand at each time period. The proposed approach compares favorably with a state-of-the-art optimization engine for very large-scale *DEM* instances (over one million 0–1 variables and three million continuous variables).

The remainder of the paper is organized as follows. Section 2 formally introduces the problem to be solved. Section 3 presents the *SDP* algorithmic framework for problem solving, and introduces the concepts to be used for obtaining the *EFV* curves. Section 4 proposes the scheme for computing them. Section 5 presents the production planning problem. Section 6 reports on the computational experience. Finally, Section 7 concludes.

2. Problem statement

Let the scenario tree shown in Fig. 1 represent the stochasticity of the problem to be dealt with. Each node in the figure represents a point in time where a decision can be made. Once a decision is made, some contingencies can arise (in this example the number of contingencies is three for time period $t = 5$), and information related to these contingencies is available at the beginning of the period. Each root-to-leaf path in the tree represents one specific scenario and corresponds to one realization of the whole set of the uncertain parameters. Each node in the tree can be associated with a scenario group, such that two scenarios belong to the same group from a given time period provided that they have the same realizations of the uncertain parameters up to that period. Accordingly with the nonanticipativity principle [8,18], the scenarios that belong to the same group at a given time period should have the same value for the related variables with the time index up to the period.

Let the following notation related to the scenario tree be used throughout the paper:

Ω set of scenarios, consecutively numbered. For instance, the path $\{1, 2, \ldots, 5, 8, 14\}$ gives a scenario, that is customarily named scenario 14.

$\mathscr{G}$ set of scenario groups, consecutively numbered.

$t(g)$ time period for scenario group g, for $g \in \mathscr{G}$.

Ω_g set of scenarios in group g, such that the scenarios that belong to the same group are identical in all realizations of the uncertain parameters up to period $t(g)$, for $g \in \mathscr{G}$ ($\Omega_g \subseteq \Omega$).

$\mathscr{K}_g$ set of scenario groups $\{k\}$ such that $\Omega_g \subseteq \Omega_k$, for $g \in \mathscr{G}$ ($\mathscr{K}_g \subset \mathscr{G}$). Notice that the (sole) ancestor path from the associated node with scenario group g to the root node in the given scenario tree passes through the