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An approximate dynamic programming approach to convex quadratic knapsack problems

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Abstract

Quadratic knapsack problem (QKP) has a central role in integer and combinatorial optimization, while efficient algorithms to general QKPs are currently very limited. We present an approximate dynamic programming (ADP) approach for solving convex QKPs where variables may take any integer value and all coefficients are real numbers. We approximate the function value using (a) continuous quadratic programming relaxation (CQPR), and (b) the integral parts of the solutions to CQPR. We propose a new heuristic which adaptively fixes the variables according to the solution of CQPR. We report computational results for QKPs with up to 200 integer variables. Our numerical results illustrate that the new heuristic produces high-quality solutions to large-scale QKPs fast and robustly.

Keywords: Approximate dynamic programming; Quadratic knapsack problem; Heuristics

0. Introduction

Quadratic knapsack problem (QKP) has a central role in integer and combinatorial optimization. QKP calls for maximizing a quadratic objective function subject to a knapsack constraint, where all coefficients in the constraint are assumed to be positive and all variables are integers.

Knapsack problems including QKP have been intensively studied since the emergence of operational research, both because of their immediate applications in industry and financial management, but more especially for theoretical reasons. From a practical point of view, a large number of problems can be formulated as QKPs (Hammer et al. [1]). Examples include selection problems (Laughhunn [2]; Gallo

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et al. [3]), compiler design problems (Johnson et al. [4]), capacity planning problems (Hua and Banerjee [5]), and so on. For an overview of several knapsack problems and their applications, see Martello and Toth [6] and Kellerer et al. [7]. Convex quadratic knapsack problems also have a variety of applications such as promotion models (McCallum [8]), capital budgeting (Mathur and Salkin [9]; Djerdjour et al. [10]), hydrological studies and location problems (Gallo et al. [3]).

From a theoretical point of view, QKP often occurs by relaxation of different integer programming problems, e.g. by surrogate relaxation of the set covering problem and complier design problem (Johnson et al. [4]; Ferreira et al. [11]). In such applications, we need to solve a knapsack problem each time, demanding extremely fast solution time. Furthermore, the study of QKP is very interesting due to the many connections with other fields of optimizations, such as graph theory and various branches of continuous optimization.

Currently, practical approaches for solving QKP are branch and bound algorithms, in which, techniques such as continuous relaxation, derivation of upper planes, linearization, reformulation, Lagrangian decomposition, semidefinite relaxation, have been employed in obtaining upper bounds for QKPs (Kellerer et al. [7]). Although branch and bound algorithms can obtain the exact solution of QKP, they are of little use if the size of QKP is large or QKP appears as a subproblem within general constrained quadratic programming (Helmberg et al. [12]).

Theoretically, knapsack problems can also be solved by dynamic programming (DP). However, this approach is usually impractical for large-scale problems because of their computation complexity and storage requirement. To solve large-scale problems, several researchers have investigated approximation in DP, which is known as approximate dynamic programming (ADP). Many methods such as function approximation, Lagrangian multiplier and successive approximation methods have been proposed to contribute diverse ADP methodologies (Bellman [13]; Morin [14]; Cooper and Cooper [15]). Recently Bertsimas and Demir [16] presented an ADP approach for the multidimensional knapsack problem and gave some promising computational results.

Some researchers exploited the graph-theoretic structure of QKPs. Billionnet [17] has shown that the cardinality-constrained quadratic 0-1 knapsack problem can be solved in $O(n)$ steps if the graph induced by the objective function is a tree. Rader Jr. and Woeginger [18] presented pseudo-polynomial time algorithms for the special case of QKP where all cost coefficients in the objective function are nonnegative integers and where the underlying graph is edge series-parallel.

As Rader Jr. and Woeginger [18] pointed out that the structure of the graph naturally defined by coefficients of the objective function plays a major role in the solution of QKP problem, we thus can divide all 0-1 QKPs into three cases: (i) Supermodular case, which requires that all objective function coefficients be integers and nonnegative, is firstly introduced by Gallo et al. [3]. (ii) The case where all objective function coefficients are integers but are with mixed signs. (iii) The case where all objective function coefficients are real numbers.

Up till now, case (i) has been extensively studied, e.g., by Billionnet [19], Caprara et al. [20], Kellerer et al. [7], and Rader Jr. and Woeginger [18]. Caprara et al. [20] reported exact solution of instances with up to 400 binary variables with density (defined by the percentage of nonzero elements in quadratic cost coefficient matrix) 25%. However, for high density (e.g. 90%), the algorithm cannot solve problems with more than 80 binary variables within the time limit of 50,000 s. Nevertheless, Caprara et al. [20] did provide the best result for case (i) we have ever seen considering both solution quality and solution time. Case (ii) has been studied by Rader Jr. and Woeginger [18] and Forrester [21]. Rader Jr. and Woeginger [18] proved that case (ii) is strongly NP-complete, without giving any efficient approximate algorithm