Journal of Mathematical Economics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Journal of Mathematical Economics

journal homepage: www.elsevier.com/locate/jmateco

Precautionary saving in the large: *n*th degree deteriorations in future income

Liqun Liu

Private Enterprise Research Center, 4231 TAMU, Texas A&M University, College Station, TX 77843, United States

ARTICLE INFO

Article history:
Received 8 July 2013
Accepted 9 December 2013
Available online xxxx

Keywords:
Precautionary saving
Precautionary premium
Stochastic dominance
Higher degree risk increase
Mean preserving stochastic dominance

ABSTRACT

Motivated by Eeckhoudt and Schlesinger's (2008) general characterization of the precautionary saving motive against *n*th degree deteriorations in future income, this note generalizes the comparative precautionary premium analysis of Kimball (1990) for 2nd degree risk increases in future income to a comparative precautionary premium analysis for *n*th degree deteriorations in future income.

1. Introduction

Leland (1968), Sandmo (1970) and Dreze and Modigliani (1972) (L–S–D–M hereafter) demonstrate that decision makers with a precautionary saving motive – those who would increase their savings if a certain future (labor) income is replaced with a random future income with the same mean – are characterized by utility functions with a positive third derivative (prudence). Because a positive third derivative also characterizes downside risk aversion (Menezes et al., 1980), one can interpret precautionary saving as coming from downside risk aversion or 3rd degree risk aversion.[1]

To quantify the strength of the precautionary saving motive, Kimball (1990) introduces the "precautionary premium", the fixed reduction in a nonrandom future income that has the same effect on saving as the addition of a mean zero risk to the future income. Kimball then establishes that, for two decision makers indicated by their second period utility functions $u_1(x)$ and $v_1(x)$ respectively,[2] $u_1(x)$ has a larger precautionary premium than $v_1(x)$ if and only if the former uniformly has a larger absolute prudence measure than the latter, or $-u_1'''/u_1'' \geq -v_1'''/v_1''$ for all x.

More recently, Eeckhoudt and Schlesinger (2008) argue that undesirable changes in the distribution of future income are not limited to the introduction of a risk, but can be risk increases of an arbitrary *n*th degree, where n is an integer such that $n \geq 1$.[3] The undesirable changes in the distribution of future income may also take the form of a deterioration in the sense of *n*th degree stochastic dominance. In the context of the wider range of deteriorations in future income, Eeckhoudt and Schlesinger find that prudence, or 3rd degree risk aversion, is no longer synonymous with the precautionary saving motive. Specifically, they generalize L–S–D–M's finding to the following: (a) a change in future income always leads to more saving for every second period utility function $u_1(x)$ that is $(n+1)$th degree risk averse if and only if the change is an *n*th degree risk increase; (b) a change in future income always leads to more saving for every second period utility function $u_1(x)$ that is *s*th degree risk averse for all $1 \leq s \leq n+1$ if and only if the change is an *n*th degree stochastically dominated one.[4] However, Eeckhoudt and Schlesinger do not compare precautionary premiums of different decision makers.

This note generalizes Kimball's comparative precautionary premium analysis for 2nd degree risk increases in future income to a comparative precautionary premium analysis for a general class of *n*th degree deteriorations in future income, where $n \geq 2$, parallel to Eeckhoudt and Schlesinger's (2008) generalization of L–S–D–M.

E-mail address: lliu@tamu.edu.

[1] For intuitive explanations of why the positive third derivative of the utility function causes the precautionary saving motive, see Menegatti (2007) and Eeckhoudt and Schlesinger (2009).

[2] It is customary in the two-period consumption model to use $u_0(x)$ for the first period utility function and $u_1(x)$ for the second period utility function.

[3] For the definition of *n*th degree risk increases, see Definition 1 in Section 2. Note that $\tilde{y}$ being a 1st degree risk increase from $\tilde{x}$ simply means $\tilde{x}$ stochastically dominating $\tilde{y}$ in the 1st degree.

[4] For the definition of *n*th degree risk aversion, see Definition 4 in Section 2. Note that $u(x)$ being 1st degree risk averse simply means $u(x)$ is increasing. Jouini et al. (2013) extend Eeckhoudt and Schlesinger's analysis by studying the effects of *n*th degree risk increases and *n*th degree risk aversion in a more general class of decision problems.

0304-4068/$ – see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.jmateco.2013.12.004

In particular, a corollary (Corollary 1 in Section 3) states that a second period utility function $u_1(x)$ has a larger precautionary premium than another second period utility function $v_1(x)$ for every nth degree risk increase in future income if and only if there exists $\lambda > 0$ such that $\frac{u_1^{(n+1)}(x)}{v_1^{(n+1)}(x)} \geq \lambda \geq \frac{u_1^{(2)}(y)}{v_1^{(2)}(y)}$ for all x and y, where $u^{(k)}(x)$ stands for the kth derivative of function $u(x)$.

Note that the above characterizing condition is stronger than Kimball's condition when $n = 2$. This is because (2nd degree) risk increases represent a larger category of changes in a random variable (which could be degenerate) than simple risk introductions to an otherwise certain outcome. Based on Liu and Meyer (2013), this characterizing condition for comparative precautionary premiums for nth degree risk increases in future income is exactly the definition of $u_1(x)$ being $((n+1)/2)$th degree Ross more risk averse than $v_1(x)$, hence is given an economically meaningful interpretation. More generally, Liu and Meyer's notion of (n/m)th degree Ross more risk aversion, Definition 6 in Section 2, is a generalization of the existing nth degree Ross more risk aversion when the strength of aversion to an nth degree risk increase is measured by the individual's willingness to pay for avoiding the nth degree risk increase in terms of accepting an mth degree risk increase instead, where $1 \leq m \leq n-1$.[5]

2. A general result on comparative risk premiums for *n*th degree deteriorations in wealth

Kimball (1990) derives his finding on comparative precautionary premiums from the existing result of Arrow (1971) and Pratt (1964) on comparative risk premiums. We follow the same strategy and establish in this section a general result on comparative risk premiums for undesirable nth degree changes in the random wealth, where $n \geq 2$.[6] The two specific types of nth degree changes from $\tilde{x}$ to $\tilde{y}$ are defined below, both of which are identical to the Rothschild and Stiglitz (1970) risk increase when $n = 2$. For notational convenience, denote by "$\succcurlyeq_n$" the standard relation of nth degree stochastic dominance.[7]

Definition 1 (*nth Degree Risk Increase*)**.** $\tilde{y}$ is an nth degree risk increase from $\tilde{x}$ if $\tilde{x} \succcurlyeq_n \tilde{y}$ and $E(\tilde{y}^j) = E(\tilde{x}^j)$ for $j = 1, \ldots, n-1$.

Definition 2 (*nth Degree Mean-Preserving Stochastic Dominance*)**.** $\tilde{x}$ dominates $\tilde{y}$ in the nth degree mean-preserving stochastic dominance if $\tilde{x} \succcurlyeq_n \tilde{y}$ and $E(\tilde{x}) = E(\tilde{y})$.

Definition 1 is originally given by Ekern (1980) and Definition 2 by Denuit and Eeckhoudt (2013). Obviously, if $\tilde{y}$ is an nth degree risk increase from $\tilde{x}$, then $\tilde{x}$ dominates $\tilde{y}$ in the nth degree mean-preserving stochastic dominance. Therefore the nth degree mean-preserving stochastic dominance places fewer restrictions on comparable pairs of random variables than the nth degree risk increase. We further propose the following definition concerning nth degree deteriorations in a random variable that is more general than both Definitions 1 and 2.

Definition 3 (*nth Degree l-MPSD*)**.** For any given integer l such that $1 \leq l \leq n-1$, $\tilde{x}$ dominates $\tilde{y}$ in the nth degree l-MPSD (first l moments preserving stochastic dominance) if $\tilde{x} \succcurlyeq_n \tilde{y}$ and $E(\tilde{x}^j) = E(\tilde{y}^j)$ for $j = 1, \ldots, l$.

In addition to the above three definitions, we also need the following definitions of nth degree risk aversion due to Ekern (1980) and nth degree Ross more risk aversion due to Jindapon and Neilson (2007). As a generalization of nth degree Ross more risk aversion, we further give the definition of (n/m)th degree Ross more risk aversion due to Liu and Meyer (2013). This final definition will be useful in giving an economic interpretation to a characterizing condition for comparative precautionary premiums in the next section.

Definition 4 (*nth Degree Risk Aversion*)**.** Utility function $u(x)$ is nth degree risk averse if $(-1)^{n+1}u^{(n)}(x) > 0$ for all x.

Definition 5 (*nth Degree Ross More Risk Aversion*)**.** For two increasing utility functions $u(x)$ and $v(x)$ that are both nth degree risk averse, $u(x)$ is nth degree Ross more risk averse than $v(x)$ if there exists $\lambda > 0$ such that $\frac{u^{(n)}(x)}{v^{(n)}(x)} \geq \lambda \geq \frac{u'(y)}{v'(y)}$ for all x, y.

Definition 6 (*(n/m)th Degree Ross More Risk Aversion*)**.** For two utility functions $u(x)$ and $v(x)$ that are each both nth degree risk averse and mth degree risk averse, $u(x)$ is (n/m)th degree Ross more risk averse than $v(x)$ if there exists $\lambda > 0$ such that $\frac{u^{(n)}(x)}{v^{(n)}(x)} \geq \lambda \geq \frac{u^{(m)}(y)}{v^{(m)}(y)}$ for all x, y.

Definition 1 is closely related to Definitions 4 and 5 as follows. Ekern (1980) shows that $\tilde{y}$ is an nth degree risk increase from $\tilde{x}$ if and only if $\tilde{x}$ is preferred to $\tilde{y}$ by all $u(x)$ that are nth degree risk averse.[8] Li (2009) and Denuit and Eeckhoudt (2010) demonstrate that $u(x)$ is nth degree Ross more risk averse than $v(x)$ if and only if $u(x)$ is always willing to pay a larger risk premium than $v(x)$ to avoid a given nth degree risk increase in the random wealth.[9] Similar relationships between Definition 2 and Definitions 4 and 5 are established by Denuit and Eeckhoudt (2013). They show that $\tilde{x}$ dominates $\tilde{y}$ in nth degree mean-preserving stochastic dominance if and only if $\tilde{x}$ is preferred to $\tilde{y}$ by all $u(x)$ that are kth degree risk averse for all $k = 2, \ldots, n$, and that $u(x)$ is kth degree Ross more risk averse than $v(x)$ for $k = 2, \ldots, n$ if and only if $u(x)$ is always willing to pay a larger risk premium than $v(x)$ to avoid a given nth degree mean-preserving stochastic dominance deterioration in the random wealth.

The relationships between Definition 3 on one side and Definitions 4 and 5 on the other are given in the following two theorems, which are generalizations of the corresponding results in the literature. The proof of Theorem 1 is straightforward from integration by parts and hence omitted.[10] The proof of Theorem 2 is given in the Appendix.

Theorem 1. *Suppose that $n \geq 2$ and $1 \leq l \leq n-1$. $\tilde{x}$ dominates $\tilde{y}$ in the nth degree l-MPSD if and only if $\tilde{x}$ is preferred to $\tilde{y}$ by all $u(x)$ that are kth degree risk averse for $k = l+1, \ldots, n$.*

Theorem 2. *Suppose that $n \geq 2$ and $1 \leq l \leq n-1$. For two increasing utility functions $u(x)$ and $v(x)$ that are both kth degree risk averse for $k = l+1, \ldots, n$, the following conditions are equivalent:*

(i) *$u(x)$ is kth degree Ross more risk averse than $v(x)$ for $k = l+1, \ldots, n$;*
(ii) *there exists $\lambda > 0$ and $\phi(x)$ such that $u = \lambda v + \phi$, where $\phi'(x) \leq 0$ and $(-1)^{k+1}\phi^{(k)}(x) \geq 0$ for all x and $k = l+1, \ldots, n$;*
(iii) *$\pi_u \geq \pi_v$ for all $\tilde{x}$ and $\tilde{y}$ such that $\tilde{x}$ dominates $\tilde{y}$ in nth degree l-MPSD, where $Eu(\tilde{x} - \pi_u) = Eu(\tilde{y})$ and $Ev(\tilde{x} - \pi_v) = Ev(\tilde{y})$.*

5 In Liu and Meyer's notation, the existing nth degree Ross more risk aversion due to Jindapon and Neilson (2007) (Definition 5) is a special case of (n/m)th degree Ross more risk aversion when $m = 1$.

6 As is well known from Arrow (1971), Pratt (1964) and Ross (1981), the comparative risk premium analysis can only be meaningfully applied to changes that do not alter the mean of the initial wealth distribution (where the risk premium is subtracted), ruling out 1st degree deteriorations in wealth.

7 For example, see Ingersoll (1987).

8 This result is a higher degree generalization of Rothschild and Stiglitz (1970), where $n = 2$, and Menezes et al. (1980), where $n = 3$.

9 This result is a higher degree generalization of Ross (1981), where $n = 2$, and Modica and Scarsini (2005), where $n = 3$.

10 See proofs of similar results in Ekern (1980) and Denuit and Eeckhoudt (2013).

Please cite this article in press as: Liu, L., Precautionary saving in the large: nth degree deteriorations in future income. Journal of Mathematical Economics (2013), http://dx.doi.org/10.1016/j.jmateco.2013.12.004