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Ripplet: A new transform for image processing

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ABSTRACT

Efficient representation of images usually leads to improvements in storage efficiency, computational complexity and performance of image processing algorithms. Efficient representation of images can be achieved by transforms. However, conventional transforms such as Fourier transform and wavelet transform suffer from discontinuities such as edges in images. To address this problem, we propose a new transform called ripplet transform. The ripplet transform is a higher dimensional generalization of the curvelet transform, designed to represent images or two-dimensional signals at different scales and different directions. Specifically, the ripplet transform allows arbitrary support c and degree d while the curvelet transform is just a special case of the ripplet transform (Type I) with $c = 1$ and $d = 2$. Our experimental results demonstrate that the ripplet transform can provide efficient representation of edges in images. The ripplet transform holds great potential for image processing such as image restoration, image denoising and image compression.

1. Introduction

Efficient representation of images or signals is critical for image processing, computer vision, pattern recognition and image compression. Harmonic analysis [1] provides a methodology to represent signals efficiently. Specifically, harmonic analysis is intended to efficiently represent a signal by a weighted sum of basis functions; here the weights are called coefficients and the mapping from the input signal to the coefficients is called transform. In image processing, Fourier transform is usually used. However, Fourier transform can only provide an efficient representation for smooth images but not for images that contain edges. Edges or boundaries of objects cause discontinuities or singularities in image intensity. How to efficiently represent singularities in images poses a great challenge to harmonic analysis. It is well known that one-dimensional (1D) singularities in a function (which has finite duration or is periodic) destroy the sparsity of Fourier series representation of the function, which is known as Gibbs phenomenon. In contrast, wavelet transform is able to efficiently represent a function with 1D singularities [2,3]. However, typical wavelet transform is unable to resolve two-dimensional (2D) singularities along arbitrarily shaped curves since typical 2D wavelet transform is just a tensor product of two 1D wavelet transforms, which resolve 1D horizontal and vertical singularities, respectively.

To overcome the limitation of wavelet, ridgelet transform [4,5] was introduced. Ridgelet transform can resolve 1D singularities along an arbitrary direction (including horizontal and vertical direction). Ridgelet transform provides information about orientation of linear edges in images since it is based on Radon transform [6], which is capable of extracting lines of arbitrary orientation.

Since ridgelet transform is not able to resolve 2D singularities, Candes and Donoho proposed the first generation curvelet transform based on multi-scale ridgelet [7,8]. Later, they proposed the second generation curvelet transform [9,10]. Curvelet transform can resolve 2D singularities along smooth curves. Curvelet transform uses a parabolic scaling law to achieve anisotropic directionality. From the perspective of microlocal analysis, the anisotropic property of curvelet transform guarantees resolving 2D singularities along C^2 curves [11,9,10,12]. Similar to curvelet, contourlet [13,14] and bandlet [15] were proposed to resolve 2D singularities.

However, it is not clear why parabolic scaling was chosen for curvelet to achieve anisotropic directionality. Regarding this, we have two questions: is the parabolic scaling law optimal for all types of boundaries? If not, what scaling law will be optimal? To address these two questions, we intend to generalize the scaling law, which results in a new transform called *ripplet transform* Type I. Ripplet transform Type I generalizes curvelet transform by adding two parameters, i.e., support c and degree d; hence, curvelet transform is just a special case of ripplet transform Type I with $c = 1$ and $d = 2$. The new parameters, i.e., support c and degree d, provide ripplet transform with anisotropy capability of representing singularities along arbitrarily shaped curves. The ripplet transform has the following capabilities:

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- *Multi-resolution*: Ripplet transform provides a hierarchical representation of images. It can successively approximate images from coarse to fine resolutions.
- *Good localization*: Ripplet functions have compact support in frequency domain and decay very fast in spatial domain. So ripplet functions are well localized in both spatial and frequency domains.
- *High directionality*: Ripplet functions orient at various directions. With the increasing of resolution, ripplet functions can obtain more directions.
- *General scaling and support*: Ripplet functions can represent scaling with arbitrary degree and support.
- *Anisotropy*: The general scaling and support result in anisotropy of ripplet functions, which guarantees to capture singularities along various curves.
- *Fast coefficient decay*: The magnitudes of ripplet transform coefficients decay faster than those of other transforms', which means higher energy concentration ability.

Note that we have also developed ripplet transform Type II and Type III, which will be described in our future work.

To evaluate the performance of ripplet transform for image processing, we conduct experiments on synthetic and natural images in image compression and denoising applications. Our experimental results demonstrate that for some images, ripplet transform can represent images more efficiently than DCT and discrete wavelet transform (DWT), when the compression ratio is high. When used for image compression, ripplet transform based image coding outperforms JPEG for the whole bit rate range; and it achieves performance comparable to JPEG2000, when the compression ratio is high; but ripplet transform can provide better visual quality than JPEG2000. Our experimental results also show that the ripplet transform achieves superior performance in image denoising.

The remainder of the paper is organized as below. In Section 2, we review the continuous curvelet transform in spatial domain and frequency domain, and analyze the relations between them. In Section 3, we generalize the scaling law of curvelet to define ripplets and introduce continuous ripplet transform and inverse continuous ripplet transform. Then we discuss the discretization of ripplet transform in Section 4. We analyze ripplet functions from the perspective of frames in Section 5. Section 6 presents experimental results that demonstrate the good properties of ripplets. Section 7 concludes this paper and points out future research directions.

2. Continuous curvelet transform

Similar to the definition of wavelets, the whole curvelet family is constructed based on the element curvelet functions. The element curvelet functions vary from coarse to fine scales. The curvelet functions are translated and rotated versions of the element functions. The 2D curvelet function is defined as below [7,8]

$$\gamma_{a\vec{b}\theta}(\vec{x}) = \gamma_{a\vec{0}0}(R_\theta(\vec{x}-\vec{b})), \tag{1}$$

where $R_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is the rotation matrix, which rotates θ radians. $\vec{x}$ and $\vec{b}$ are 2D vectors. $\gamma_{a\vec{0}0}$ is the element curvelet function.

The element curvelet function $\gamma_{a\vec{0}0}$ with scale parameter a is defined in the frequency domain in polar coordinates [8].

$$\hat{\gamma}_a(r,\omega) = a^{3/4}W(a\cdot r)V(\omega/\sqrt{a}), \tag{2}$$

where $\hat{\gamma}_a(r,\omega)$ is the Fourier transform of $\gamma_{a\vec{0}0}$ in polar coordinate system. $W(r)$ is a 'radial window' and $V(\omega)$ is an 'angular window'. These two windows have compact supports on [1/2,2] and [−1,1], respectively. They satisfy the following admissibility conditions:

$$\int_{1/2}^{2} W^2(r)\frac{dr}{r} = 1, \tag{3}$$

$$\int_{-1}^{1} V^2(t)dt = 1. \tag{4}$$

These two windows partition the polar frequency domain into 'wedges' shown in Fig. 1.

From Eqs. (2)–(4), we know that the Fourier transform of curvelet function has a compact support in a small region which is the Cartesian product of $r \in \left[\frac{1}{2a}, \frac{2}{a}\right]$ and $\omega \in \left[-\sqrt{a}, \sqrt{a}\right]$. Curvelet also has small effective regions and decays rapidly in spatial domain. Compared to wavelets, in addition to the scaling information and position information, curvelet functions have another parameter to represent directional information. An intuitive way to obtain direction information is using rotated wavelet. However, the isotropic property of rotated wavelet transform makes the rotation unsuitable for resolving the wavefront set [9,10]. The parabolic scaling used in the definition of curvelet functions guarantees the effective length and width of the region to satisfy: $width \approx length^2$ and leads to anisotropic behavior of curvelets, which makes curvelet transform suitable for resolving arbitrary wavefront. The parabolic scaling is the most important property of curvelet transform and also the key difference between the curvelet and the rotated wavelet.

Given a 2D integrable function $f(\vec{x})$, the continuous curvelet transform is defined as the inner product of $f(\vec{x})$ and the curvelet function [9,10,16]

$$C(a,\vec{b},\theta) = \langle f, \gamma_{a\vec{b}\theta}\rangle = \int f(\vec{x})\overline{\gamma_{a\vec{b}\theta}(\vec{x})}d\vec{x}, \tag{5}$$

where $C(a,\vec{b},\theta)$ are the curvelet coefficients and $\overline{(\cdot)}$ denotes the conjugate operator. The curvelet coefficients describe the characteristics of signal at various scales, locations and directions.

In fact, the curvelet transform only captures the characteristics of high frequency components of $f(\vec{x})$, since the scale parameter a cannot take the value of infinity. So the 'full' continuous curvelet transform consists of fine-scale curvelet transform and coarse-scale isotropic wavelet transform. The 'full' curvelet transform is invertible. We can perfectly reconstruct the input function based on its curvelet coefficients. With the 'full' curvelet transform, the Parseval formula holds [9,10,16]. If $f(\vec{x})$ is a high-pass function, it can be reconstructed from the coefficients obtained from Eq. (5) through

$$\tilde{f}(\vec{x}) = \int C(a,\vec{b},\theta)\gamma_{a\vec{b}\theta}(\vec{x})da\,d\vec{b}\,d\theta/a^3 \tag{6}$$

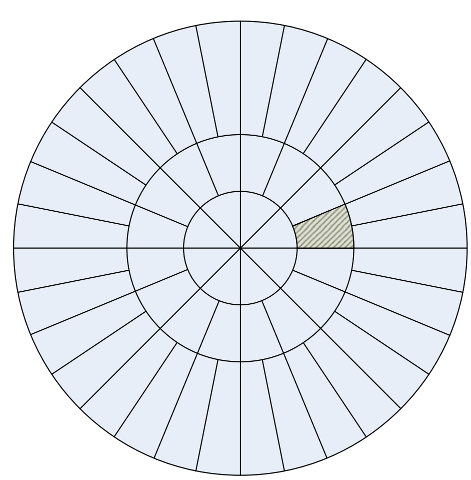

Fig. 1. The tiling of polar frequency domain. The shadowed 'wedge' corresponds to the frequency transform of the element function.