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Continuous Optimization

# Robust solutions to multi-objective linear programs with uncertain data ☆

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ABSTRACT

In this paper we examine multi-objective linear programming problems in the face of data uncertainty both in the objective function and the constraints. First, we derive a formula for the radius of robust feasibility guaranteeing constraint feasibility for all possible scenarios within a specified uncertainty set under affine data parametrization. We then present numerically tractable optimality conditions for minmax robust weakly efficient solutions, i.e., the weakly efficient solutions of the robust counterpart. We also consider highly robust weakly efficient solutions, i.e., robust feasible solutions which are weakly efficient for any possible instance of the objective matrix within a specified uncertainty set, providing lower bounds for the radius of highly robust efficiency guaranteeing the existence of this type of solutions under affine and rank-1 objective data uncertainty. Finally, we provide numerically tractable optimality conditions for highly robust weakly efficient solutions.

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## 1. Introduction

Consider the deterministic multi-objective linear programming problem

$$(\overline{P}) \qquad \text{V-min}\{(\overline{c}_1^\top x, \ldots, \overline{c}_m^\top x) : \overline{a}_j^\top x \geq \overline{b}_j, \ j \in J\},$$

where V- min stands for *vector minimization*, $\overline{c}_i \in \mathbb{R}^n$ (interpreted as a column vector) for $i \in I := \{1, \ldots, m\}$, the symbol $^\top$ denotes transpose, $x \in \mathbb{R}^n$ is the decision variable, and $(\overline{a}_j, \overline{b}_j) \in \mathbb{R}^n \times \mathbb{R}$, for $j \in J := \{1, \ldots, p\}$, are the constraint input data of the problem. The real $m \times n$ matrix $\overline{C}$ whose rows are the vectors $\overline{c}_i$, $i \in I$, is called the *objective matrix*. The problem $(\overline{P})$ has been extensively studied in the literature (see, e.g., the overviews Branke, Deb, Miettinen, & Slowinski, 2008; Ehrgott, 2005), where perfect information is often assumed (that is, accurate values for the input quantities or parameters), despite the reality that such precise knowledge is rarely available in practice for real-world optimization problems.

The data of real-world optimization problems are often uncertain (that is, they are not known exactly at the time of the decision) due to estimation errors, prediction errors, or lack of information. Scalar uncertain optimization problems have been traditionally treated via sensitivity analysis which estimates the impact of small perturbations of the data in the optimal value, while robust optimization, which provides a deterministic framework for uncertain problems, has recently emerged as a powerful alternative approach (see, for instance, Ben-Tal & El Ghaoui, 2009; Bertsimas & Sim, 2004; El Ghaoui & Lebret, 1997; Goberna, Jeyakumar, Li, & López, 2013; Jeyakumar & Li, 2010).

Particular types of uncertain multi-objective linear programming problems have already been studied, e.g., Sitarz (2008) considers changes in one objective function via sensitivity analysis, while Pando, Luc, and Pardalos (2013) consider changes in the whole objective function $x \mapsto \overline{C}x$ and Goberna, Jeyakumar, Li, and Vicente-Pérez (2014) considers change in the constraints by using different robustness approaches. The purpose of the present work is to study multi-objective linear programming problems in the face of data uncertainty both in the objective function and constraints from a robustness perspective.

Following the robust optimization framework, the multi-objective problem $(\overline{P})$ in the face of *data uncertainty* both in the objective matrix and in the data of the constraints can be captured by a parameterized multi-objective linear programming problem of the form

$$(P) \qquad \text{V-min}\{(c_1^\top x, \ldots, c_m^\top x) : a_j^\top x \geq b_j, \ j \in J\}$$

where the input data, $c_i$, $i \in I$, and $(a_j, b_j)$, $j \in J$, are uncertain vectors, $C := (c_1, \ldots, c_m) \in \mathcal{U} \subset \mathbb{R}^{n \times m}$ and $(a_j, b_j) \in \mathcal{V}_j \subset \mathbb{R}^{n+1}$, $j \in J$ and the sets $\mathcal{U}$ and $\mathcal{V}_j$, $j \in J$, are specified uncertainty sets that are bounded, but often infinite sets. By enforcing the constraints for all possible uncertainties within $\mathcal{V}_j$, $j \in J$, the uncertain problem becomes the uncertain multi-objective linear semi-infinite programming problem

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$$\text{V-}\min\{(c_1^\top x, \ldots, c_m^\top x) : a_j^\top x \geq b_j,\ \forall (a_j, b_j) \in \mathcal{V}_j,\ j \in J\}, \quad (1)$$

where $(c_1, \ldots, c_m) \in \mathcal{U}$ and whose feasible set,

$$X := \{x \in \mathbb{R}^n : a_j^\top x \geq b_j,\ \forall (a_j, b_j) \in \mathcal{V}_j,\ j \in J\}, \quad (2)$$

is called *robust feasible set* of $(P)$ and $x \in X$ is called a *robust feasible solution*.

Following the recent work on robust linear programming (see Ben-Tal & El Ghaoui, 2009), some of the key questions of multi-objective linear programming under data uncertainty include:

I. (*Guaranteeing robust feasible solutions*) How to guarantee non-emptiness of the robust feasible set $X$ for specified uncertainty sets $\mathcal{V}_j$, $j \in J$?
II. (*Guaranteeing and identifying robust efficient solutions*) Which robust feasible solutions of $(P)$ are robust efficient solutions (see the paragraph below) that are immune to objective data uncertainty and what are the mathematical characterizations that identify robust efficient solutions? How to guarantee the existence of robust efficient solutions?
III. (*Numerical tractability of robust efficient solutions*) For what specified classes of uncertainty sets $\mathcal{U}$ and $\mathcal{V}_j$, $j \in J$, the robust efficient solution characterizations can be numerically checked using existing multi-objective programming techniques?

In this paper, we provide some answers to the above questions for the uncertain multi-objective linear programming problem $(P)$ in the face of data uncertainty by focusing on two choices of the robust optimal solutions: the first one is called a *minmax robust efficient solution* or simply *robust efficient solution* following the approach widely used in robust scalar optimization problem (see also Ehrgott, Ide, and Schöbel (2014); Kuroiwa and Lee (2012) for recent development), and corresponds to an efficient solution to a deterministic worst-case (minmax) multi-objective optimization problem; the second one is called *highly robust efficient solution* as in Ide and Schöbel (2013) and Kuhn, Raith, Schmidt, and Schöbel (2013) (see also Sitarz, 2008; Pando et al., 2013, Section 4), and consists of the preservation of the efficiency for all $(c_1, \ldots, c_m) \in \mathcal{U}$. So, the existence of this type of solution implies that the uncertainty set $\mathcal{U}$ is small in some sense (e.g., Cartesian products of balls in $\mathbb{R}^n$ or segments in $\mathbb{R}^{n\times m}$ emanating from some fixed data $(\bar{c}_1, \ldots, \bar{c}_m)$). To compensate the "smallness" of the uncertainty set, we focus our analysis on the larger class of highly robust solutions: highly robust weakly efficient solutions. On the other hand, in (Pando et al., 2013, Section 4), highly robust efficient solutions are considered instead of highly robust weakly efficient solutions. For the convenience of the reader, other notions of robust solutions are summarized in Appendix A.

Our key contributions are outlined as follows:

(1) We first introduce the concept of radius of robust feasibility in Section 3, guaranteeing non emptiness of the robust feasible set $X$ of $(P)$ under affinely parameterized data uncertainty. This concept is inspired by the notion of consistency radius used in linear semi-infinite programming in order to guarantee the feasibility of the nominal problem under perturbations preserving the number of constraints (Cánovas, López, Parra, & Toledo, 2005, Cánovas, López, Parra, & Toledo, 2011). We derive a formula for the effective computation of the radius of robust feasibility that also applies to single-objective linear programming under the same type of uncertainty.
(2) We examine the robust weakly efficient solution of an uncertain multi-objective linear programming problem in Section 4, and establish numerically tractable mathematical characterizations of robust weakly efficient solutions under various data uncertainty.
(3) We present, in Section 5, an explicit formula for the radius of highly robust efficiency, i.e., the greatest value of certain parameter associated with two families of uncertainty sets for the objective data such that the corresponding multi-objective linear programming problems have highly robust weakly efficient solutions. The mentioned families are formed by Cartesian products of balls in $\mathbb{R}^n$ and by segments in $\mathbb{R}^{m\times n}$ in the direction of rank-1 matrices (the same type of uncertainty considered in Pando et al., 2013, Section 4). Recall that rank-1 matrices are the products of non-zero column vectors by non-zero row vectors (see Osnaga, 2005 for other characterizations). These matrices are frequently used in computational algebra (as building blocks for more complex matrices), in conic optimization (as the rank-1 matrices are the extreme rays of the semidefinite cone), and in statistics (as the singular value decomposition gives the best rank-1 approximation of a given matrix with respect to the Frobenius and the spectral norms).
(4) We finally provide, in Section 6, numerically tractable mathematical characterizations of highly robust weakly efficient solutions under various data uncertainty.

## 2. Preliminaries

We begin this section introducing the necessary notation and concepts on multi-objective linear programming. We denote by $0_n$ and $\|\cdot\|$ the vector of zeros and the Euclidean norm in $\mathbb{R}^n$, respectively. The closed unit ball and the distance associated to the above norm are denoted by $\mathbb{B}_n$ and $d$, respectively. Given $Z \subset \mathbb{R}^n$, $\operatorname{int} Z$, $\operatorname{cl} Z$, $\operatorname{bd} Z$, and $\operatorname{conv} Z$ denote the interior, the closure, the boundary and the convex hull of $Z$, respectively, whereas $\operatorname{cone} Z := \mathbb{R}_+ \operatorname{conv} Z$ denotes the convex conical hull of $Z \cup \{0_n\}$. For $x, y \in \mathbb{R}^m$, we write $x \leq y$ $(x < y)$ when $x_i \leq y_i$ $(x_i < y_i$, respectively) for all $i \in I$. The simplex $\Delta_m$ in the space of criteria $\mathbb{R}^m$ is defined as $\Delta_m := \{\lambda \in \mathbb{R}^m_+ : \sum_{i=1}^m \lambda_i = 1\}$.

The following known dual characterizations of solutions of semi-infinite linear inequality systems play key roles in the next section in developing radius of robust feasibility formulae.

**Lemma 1** (Goberna & López, 1998, Corollaries 3.1.1 and 3.1.2)**.** *Let $T$ be an arbitrary index set. Then, $\{x \in \mathbb{R}^n : u_t^\top x \geq v_t, t \in T\} \neq \emptyset$ if and only if $(0_n, 1) \notin \operatorname{cl}\operatorname{cone}\{(u_t, v_t) : t \in T\}$. In that case, $u^\top x \geq v$ holds for any $x \in \mathbb{R}^n$ such that $u_t^\top x \geq v_t$, $\forall t \in T$, if and only if*

$$(u, v) \in \operatorname{cl}\{\operatorname{cone}\{(u_t, v_t) : t \in T\} + \mathbb{R}_+(0_n, -1)\}. \quad (3)$$

We now apply Lemma 1 to the robust feasible set

$$X := \{x \in \mathbb{R}^n : a_j^\top x \geq b_j,\ \forall (a_j, b_j) \in \mathcal{V}_j, j \in J\}.$$

**Proposition 2** (Feasibility and polyhedrality of $X$)**.** *Let $X$ be as in (2). Then the following statements hold:*

(i) *$X \neq \emptyset$ if and only if $(0_n, 1) \notin \operatorname{cl}\operatorname{cone}\{\cup_{j\in J}\mathcal{V}_j\}$.*
(ii) *If $X \neq \emptyset$ and the uncertainty sets $\mathcal{V}_j$ are all polyhedral sets, then $X$ is a polyhedral set too.*

**Proof.** (i) It is a straightforward consequence of Lemma 1.

(ii) Assume that $X \neq \emptyset$. If the uncertainty sets are polyhedral, we can write $\mathcal{V}_j = \operatorname{conv}\mathcal{E}_j + \operatorname{cone}\mathcal{D}_j$, with $\mathcal{E}_j$ and $\mathcal{D}_j$ finite sets, for each $j \in J$. Since the cone in (3) is

$$\operatorname{cl}\{\operatorname{cone}\{\cup_{j\in J}\mathcal{V}_j\} + \mathbb{R}_+(0_n, -1)\} = \operatorname{cone}\{\cup_{j\in J}(\mathcal{E}_j \cup \mathcal{D}_j)\} + \mathbb{R}_+(0_n, -1)$$

and, by the separation theorem, two non-empty closed convex sets coincide if and only if they have the same linear consequences, we have

$$X = \{x \in \mathbb{R}^n : a^\top x \geq b, (a, b) \in \cup_{j\in J}(\mathcal{E}_j \cup \mathcal{D}_j)\}.$$

Hence, the conclusion follows. □

Concerning Proposition 2, if the uncertainty set $\mathcal{V}_j$ contains no line, then $\mathcal{E}_j$, defined as in the proof of (ii) of Proposition 2, is the set of extreme points of $\mathcal{V}_j$. In particular, if $\mathcal{V}_j$ is a compact convex set for each $j \in J$ and the *strict robust feasibility condition*

$$\{x \in \mathbb{R}^n : a_j^\top x > b_j,\ \forall (a_j, b_j) \in \mathcal{V}_j, j \in J\} \neq \emptyset \quad (4)$$

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