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A linear programming-based algorithm for the signed separation of (non-smooth) convex bodies

G. Johnson [a,*], M. Ortiz [a], S. Leyendecker [b]

[a] Graduate Aerospace Laboratories, California Institute of Technology 1200 E. California Blvd., Pasadena, CA 91125, USA
[b] Chair of Applied Dynamics, University of Erlangen-Nuremburg Konrad-Zuse-Str. 3/5, D-91052 Erlangen, Germany

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ABSTRACT

A subdifferentiable global contact detection algorithm, the Supporting Separating Hyperplane (SSH) algorithm, based on the signed distance between supporting hyperplanes of two convex sets is developed. It is shown that for polyhedral sets, the SSH algorithm may be evaluated as a linear program, and that this linear program is always feasible and always subdifferentiable with respect to the configuration variables, which define the constraint matrix. This is true regardless of whether the program is primal degenerate, dual degenerate, or both. The subgradient of the SSH linear program always lies in the normal cone of the closest admissible configuration to an inadmissible contact configuration. In particular if a contact surface exists, the subgradient of the SSH linear program is orthogonal to the contact surface, as required of contact reactions. This property of the algorithm is particularly important in modeling stiff systems, rigid bodies, and tightly packed or jammed systems.

1. Introduction

The objective of this paper is to develop a contact detection algorithm and contact potential for non-smooth convex bodies. The proposed contact detection algorithm can be concisely described as a supporting separating hyperplane (SSH) test for interpenetration, and is based on standard separation theorems for compact convex sets. We develop this test in detail for polyhedral sets, where the SSH test can be effectively reformulated as a linear programming problem–the SSH LP. We further show that the subgradient of the SSH LP can be readily evaluated and that it supplies the force system at the time of contact.

1.1. Previous work

A large body of literature exists on the efficient detection of collisions, driven in large part by advances in computational geometry, computer graphics and robotics [1–13]. While several advances have taken place in the past several years, the 2001 review by Jimenez and others [14] and the 2005 book by Ericson [11] effectively summarize the essential state of the art. We follow the same track as much of this literature, which develops collision detection algorithms for use with convex polyhedra and for solid models that have been discretized by polyhedra. However, we note that the collision detection algorithm presented here is general enough to extend to an interpenetration test between any two smooth or non-smooth convex bodies, but the linear programming solution methodology is not, in general, extensible to these situations.

The most popular software packages make use of hierarchical volume bounding to organize oriented bounding bodies (OBB's) and axis-aligned bounding bodies (AABB's) into rapidly searchable data structures (Octrees, K-D trees etc.) [6,7,11]. Intended for discretized surfaces, these algorithms can quickly compute candidate areas for contact, and refine those areas to determine which simplices are actually involved in a collision. However, when the objects get close enough for the bounding volumes to suggest that contact might have taken place, a finer detection test must be used to conclusively declare that a collision has taken place. Another option that works on the coarse and fine levels, proposed by Chung and Wang, detects collision based on the existence (or not) of a separating vector [9]. While our interpenetration function is universal and robust, in that it can be evaluated for any pair of convex bodies, and is certainly capable of serving as a collision detection test, we would rather suggest its use as a final test in conjunction with one of the coarser tests referenced in this paragraph.

The interpenetration function here can be seen as an alternative to the heuristic search for a separating vector developed by Chung and Wang [9], and also as an alternative to other linear programming approaches such as those proposed by Akgunduz and others [10] and Aliyu and Al-Sultan [8], to which Seidel [15] made key contributions. The advantage of our proposed linear programming

* Corresponding author. Address: 1200 E. California Blvd., MC 105-50, Pasadena, CA 91125, USA.
E-mail address: gwen@caltech.edu (G. Johnson).

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approach to collision detection is that it provides extremely useful additional information for physics-based dynamics simulations and closest-point projection (CPP) operations. In the present work, we follow a well known path to collision detection and undertake a search for a separating vector. However, we search for this vector in an optimal way so that it always lies in the normal cone of the closest admissible configuration to an inadmissible contact configuration and respects any symmetries present in the geometry of the contact configuration. Furthermore, unlike previously proposed methods, our approach ensures that the subgradient of the SSH linear program is only non-zero with respect to degrees of freedom directly involved in the collision and also respects the geometry of the contact configuration. For example, if a contact surface exists, the subgradient is orthogonal to the contact surface.

1.2. Motivation

The aforementioned *shape useful additional information* is the key motivation for this work. By way of illustration, we may consider a widely accepted treatment of contact in the equations of motion: the introduction of a contact potential into the action functional. This potential takes the form of the indicator function, $I_{\mathcal{A}}$ of a set $\mathcal{A} \subset \mathcal{Q}$ containing all admissible (non-interpenetrating) configurations $\boldsymbol{x}$. Here, $\mathcal{Q}$ is a configuration manifold and $T_{\boldsymbol{x}}\mathcal{Q}$ is the tangent manifold to $\mathcal{Q}$ at $\boldsymbol{x}$, i.e. the state variables consist of configurations $\boldsymbol{x} \in \mathcal{Q}$ and velocities $\dot{\boldsymbol{x}} \in T_{\boldsymbol{x}}\mathcal{Q}$ [16–22]. For simplicity, we will associate both $\mathcal{Q}$ and $T_{\boldsymbol{x}}\mathcal{Q}$ with $\mathbb{R}^n$. Admissible (non-contact) configurations for $\boldsymbol{x}$ occupy the subset $\mathcal{A} \subset \mathcal{Q}$.

In the absence of other potentials and external forces, the action functional reads

$$\mathcal{I}(\boldsymbol{x}) = \int_0^T \mathcal{L}(\boldsymbol{x}, \dot{\boldsymbol{x}})dt, \tag{1}$$

for the Lagrangian

$$\mathcal{L}(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \dot{\boldsymbol{x}}^T\boldsymbol{M}\dot{\boldsymbol{x}} - I_{\mathcal{A}}(\boldsymbol{x}), \tag{2}$$

where M is an appropriate mass matrix and

$$I_{\mathcal{A}}(\boldsymbol{x}) = \begin{cases} 0 & \text{if } \boldsymbol{x} \in \mathcal{A} \\ \infty & \text{otherwise} \end{cases}. \tag{3}$$

The equations of motion can be recovered by requiring stationarity of $\mathcal{I}$

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \partial I_{\mathcal{A}}(\boldsymbol{x}) \ni \boldsymbol{0}. \tag{4}$$

In (4), $\partial I_{\mathcal{A}}(\boldsymbol{x})$ denotes the *generalized differential* of the indicator function (c.f. [16,17]). It is readily shown that the generalized differential of the indicator function of a set is given by the normal cone, $N_{\mathcal{A}}$, of the set

$$\partial I_{\mathcal{A}}(\boldsymbol{x}) = N_{\mathcal{A}}(\boldsymbol{x}). \tag{5}$$

It follows from (4) that the contact forces $\boldsymbol{f}_{con}$ are related to the normal cone by:

$$\boldsymbol{f}_{con} \in -N_{\mathcal{A}}(\boldsymbol{x}). \tag{6}$$

The normal cone is defined precisely in Section 2 of this paper. For our introductory example it is sufficient to understand that (6) is a statement that the contact forces must be orthogonal to a contact surface in an admissible configuration. Alternatively, we may consider the contact time as an additional variable, leading to the action functional [19,20]

$$\mathcal{I}(\boldsymbol{x}, t_c) = \int_0^{t_c} \mathcal{L}(\boldsymbol{x}, \dot{\boldsymbol{x}})dt + \int_{t_c}^T \mathcal{L}(\boldsymbol{x}, \dot{\boldsymbol{x}})dt. \tag{7}$$

Where $\mathcal{L}(\boldsymbol{x}, \dot{\boldsymbol{x}})$ is the same as the expression in (2). In this case, the equations of motion at the time of contact read as jump conditions on the change of momentum $\boldsymbol{p} = \boldsymbol{M}\dot{\boldsymbol{x}}$ and kinetic energy during the collision

$$\left[\boldsymbol{p}^T\boldsymbol{M}^{-1}\boldsymbol{p}\right]_{t_{c-}}^{t_{c+}} = 0 \tag{8a}$$

$$[\boldsymbol{p}]_{t_{c-}}^{t_{c+}} \in N_{\mathcal{A}}(\boldsymbol{x}(t_c)). \tag{8b}$$

Eqs. (8a) and (8b) describe the conservation of energy and momentum during the collision, respectively. In practice, the restriction that the forces from (6) and the change in momentum in (8a) be in the normal cone of the admissible set are accomplished by constraining the configuration variables to be in $\mathcal{A} \subset \mathcal{Q}$ via an interpenetration function $g(\boldsymbol{x})$ that is negative if two bodies are not overlapping and positive if they are, that is $\mathcal{A} = \{\boldsymbol{x} \in \mathcal{Q} | g(\boldsymbol{x}) \leqslant 0\}$. For example, consider a point mass in two dimensions falling onto a flat surface coincident with the x_1−axis (see Fig. 1). In this case, the admissible set of configurations for the mass are described by $\mathcal{A} = \{\boldsymbol{x} \in \mathbb{R}^2 | \langle -\hat{\boldsymbol{n}}, \boldsymbol{x}\rangle \leqslant 0\}$ with $\boldsymbol{x} = (x_1, x_2)$ and $\hat{\boldsymbol{n}} = (0, 1)^T$, and the normal cone of the admissible set has the unique values $N_{\mathcal{A}}(\boldsymbol{x}) = -\hat{\boldsymbol{n}}$ if $x_2 = 0$, $N_{\mathcal{A}}(\boldsymbol{x}) = \boldsymbol{0}$ if $x_2 > 0$ and $N_{\mathcal{A}}(\boldsymbol{x}) = \emptyset$ otherwise.

In this case, (8a) can be expressed as [20]

$$[\boldsymbol{p}]_{t_{c-}}^{t_{c+}} = \lambda \nabla g, \tag{9}$$

which for our example is equal to

$$[\boldsymbol{p}]_{t_{c-}}^{t_{c+}} = -\lambda \hat{\boldsymbol{n}} \tag{10}$$

if $x_2 = 0$. Here, $\lambda \in \mathbb{R}$ is a scalar parameter (see [20] for details). The simplest re-expression of (6) is accomplished through a smooth approximation of the indicator function

$$I_{\mathcal{A}}(\boldsymbol{x}) \approx V_{\mathcal{A}}(\boldsymbol{x}) = \begin{cases} 0 & \text{if } g(\boldsymbol{x}) < 0 \\ \frac{C}{2} g(\boldsymbol{x})^2 & \text{otherwise} \end{cases}, \tag{11}$$

where $C \in \mathbb{R} \gg 0$ is a constant. This leads to the formulation

$$\boldsymbol{f}_{con} = -\nabla V_{\mathcal{A}}(\boldsymbol{x}), \tag{12}$$

which for the point-mass example is

$$\boldsymbol{f}_{con} = \begin{cases} \boldsymbol{0} & \text{if } g(\boldsymbol{x}) < 0 \\ -C\langle -\hat{\boldsymbol{n}}, \boldsymbol{x}\rangle \hat{\boldsymbol{n}} & \text{otherwise} \end{cases}. \tag{13}$$

This simple example lends itself to a straightforward geometric interpretation of $N_{\mathcal{A}} = \partial I_{\mathcal{A}}$, in that the contact forces must be normal to the contact surface (see (12)) and that the change of momentum must also be normal to the surface (see (9)). Thus, to accurately conserve momentum and approximate the continuous equations of motion, a constraint function $g(\boldsymbol{x})$ should have the property that $\nabla g(\boldsymbol{x}) \approx N_{\mathcal{A}}(\boldsymbol{x})$, as in the point-mass example.

Some time integration schemes based on Lagrangian mechanics use a contact potential in the action functional and require a continuous interpenetration function that is at least sub-differentiable [19–23]. Such a potential is easy to construct for models of geometrically simple bodies and admissible configurations, but a good choice for this potential is much less obvious for complex geometries.

The preferred function for use in these applications has been a test for overlapping oriented simplices (OOS); i.e. tetrahedra in

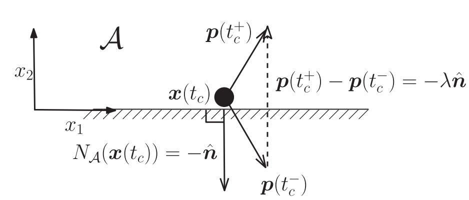

Fig. 1. A point mass striking a flat, frictionless sruface in the absence of external forces and potentials. In this example, $\lambda = 2\langle -\hat{\boldsymbol{n}}, \boldsymbol{p}(t_c^-)\rangle$.