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On the relationship between fuzzy description logics and many-valued modal logics

Marco Cerami [a], Francesc Esteva [b], Àngel García-Cerdaña [c,*]

[a] Departamento de Matemática, Universidade Federal da Bahia (UFBA), PAv Adhemar de Barros s/n, 40170-110, Salvador - BA, Brazil
[b] Artificial Intelligence Research Institute, Spanish National Research Council (IIIA-CSIC), Campus UAB, 08193 Bellaterra, Catalonia, Spain
[c] Artificial Intelligence Research Institute (IIIA-CSIC), Department of Information and Communication Technologies, University Pompeu Fabra, Tànger 122-140, 08018 Barcelona, Catalonia, Spain

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ABSTRACT

In this paper we study the relationships between a family of $\mathcal{ALC}$-like fuzzy description logics (FDLs) defined over left-continuous t-norms and some many-valued multi-modal logics (MMLs). We analyze these relationships in both directions, that is, how to merge FDLs into MMLs and vice-versa. The analysis starts from the relationships between the languages to reach systematically the deeper level of reasoning tasks. At this level we are able to truly investigate both formalisms from each other point of view. Finally, the results concerning translations between reasoning tasks are applied in order to get decidability and complexity bounds.

1. Introduction

Description Logics (DLs) are knowledge representation languages built on the basis of classical logic [2]. DLs allow the creation of knowledge bases and provide ways to reason on the contents of these bases. Even though DLs have been initially born for practical purposes, it was pointed out soon that they have a strong relationship to well-established logical formalisms. Indeed, direct translation from the DL language $\mathcal{ALC}$ into first-order and multi-modal classical logics were soon established. Basic references are [51] and [7] for the translation from DL languages into first-order logic, and [52] for the proof of the fact that $\mathcal{ALC}$ is a notational variant of the multi-modal classical logic $\mathbf{K}_m$. The relationships between DLs and first-order and modal classical logic allow us to take advantage of the big amount of computational methods and results from classical logic in benefit of DLs.

Fuzzy Description Logics (FDLs) are natural generalizations of Description Logics to the fuzzy framework. The first attempts to define FDLs were based on the interpretation of atomic concepts as fuzzy sets and atomic roles as fuzzy relations; constructors of complex concepts as the corresponding operations between fuzzy sets and fuzzy relations; and universal and existential quantifiers using infimum and supremum (see for instance [5,50,53,54,56,59]). During the 80's (see, for example [57]), the use of a t-norm as a semantics for the conjunction, and its residuum as a semantics for the implication, provided a mathematically well-founded background for fuzzy set theory. At the end of the 90's P. Hájek [32] defined what nowadays is known as Mathematical Fuzzy Logic (MFL). He defines what he calls the kernel of fuzzy logic in narrow sense as many-valued logics where the semantical counterpart is based on the residuated structures defined in the real interval

* Corresponding author.
E-mail addresses: marco.cerami@ufba.br (M. Cerami), esteva@iiia.csic.es (F. Esteva), angel@iiia.csic.es (À. García-Cerdaña).

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$[0,1]$ by t-norms and their residua. An excellent reference for the field of Mathematical Fuzzy Logic is the handbook [27]. Nowadays there exist in the literature several works on FDLs from the point of view of Mathematical Fuzzy Logic. This point of view for dealing with FDLs was proposed firstly by Hájek (see [33–35]) and recently has been also developed in [24,30] where some lines of research in this direction are proposed. The main idea is to interpret FDL formulas into first-order fuzzy logics in a similar way as formulas in classical DLs are interpreted into first-order classical logic (see [51]). A basic reference for FDLs defined from a MFL point of view is [4] (and references therein). In all these works the underlying fuzzy logic is semantically defined by interpretations on an algebra defined on the real unit interval by a continuous t-norm and its residuum.

Many-valued modal logics appeared in the literature for different reasoning modeling purposes. Fitting introduces in [28, 29] a modal logic valued on finite Heyting algebras. After this work, several many-valued modal logics have been considered in the literature from the point of view of MFL, but in many cases they are logics located at the top part of the hierarchy of modal logics where the modality is S5-like (see [32,36,37]). For a crisp accessibility relation semantics, in [38] the modal extensions of Łukasiewicz and finite-valued Łukasiewicz logics is studied, and in [58] an axiomatization of fuzzy modal logic over the standard product algebra when we have the Delta operator and rational truth constants in the language is given. More interesting for our work are [18,19,47] dealing with Gödel modal logic and the paper [16], where the minimal many-valued modal logic over a finite residuated lattice is studied in a systematic way. Our framework is also the same as the one considered in [49], where some axiomatic extensions of the minimal many-valued modal logic over a left-continuous t-norm are studied. In [14] the computational complexity of the finite-valued Łukasiewicz modal logic is established.

The main goal of this paper is to explicitly give the translations from the description language called $\mathcal{IALCE}$ and their sublanguages into the corresponding many-valued multi-modal logics (MMLs). This paper focusses on the language $\mathcal{IALCE}$ and its comparison to the minimal modal logic of a left-continuous t-norm $*$, defined semantically as the logic of all the Kripke frames whose accessibility relations are evaluated on the algebra defined on $[0,1]$ by $*$ and its residuum. Far from being a limitation, the focus of this work is chosen for its generality. The language $\mathcal{IALCE}$ is chosen because it is a notational variant of the minimal modal logic semantically defined over a t-norm and its residuum. We also consider this logic expanded with a global modality and truth constants, since this allows the translation of inclusion axioms in a similar way than in the classical case.

The main motivation of works dealing with the relation between classical DLs and classical first-order and modal logics was transferring results and methods from one field to the other, above all under a computational point of view. This motivation is not so important in the case of FDLs and many-valued modal logics since: *a*) there are less results in first-order fuzzy logic than in the classical framework; and *b*) the field of modal fuzzy logic is even less studied. Nevertheless, we think that a comparison between both formalisms is interesting despite the early stage of the research in both fields, since it may provide a deeper insight for both formalisms as well as novel ideas and methods. In our work we do not translate only the FDL syntax, but try to make a confrontation between the respective reasoning systems. The idea is that the reasoning systems are the core of a reasoning formalism and the translation of the syntax should be considered just as an introductory step towards their comparison. Indeed, our approach goes beyond than the one in [52], where just a translation between the respective syntaxes is provided. At the time of [52] knowledge bases were not considered, so a translation of the syntax was enough. Nowadays the reasoning systems of DLs and modal logics became more complex. They also become inherently different and can be barely translated into each other. Hence, defining a translation of the syntax is not anymore enough. From [52] one has the impression that DLs can be entirely translated into ML, but the possibility of obtaining computability results must be grounded on the translation of the reasoning tasks, not just on the possibility of translating the syntax.

In [3] it is also considered the problem of translating DL reasoning tasks into logical problems in classical multi-modal logic. But still in [3] the translation is though as one-way directional: from logical problems into DL reasoning tasks. In the current work, we provide a systematic and general framework where not only the syntax, but also what the system should compute is translated, when it is possible. In this sense, our work is barely comparable with the classical results. The framework that we propose in the present work is more general than the classical ones not only because it focusses on more general logical formalisms, but also, and specifically, because the focus is on the reasoning core of the respective formalisms, rather than on the syntaxes. For all the above, this work can be considered as an initial systematic settlement suitable of further developments. Nevertheless, we are already able to provide a general landscape of the relationship between FDLs and FML that is much more extensive than the one provided in the classical framework.

The paper is organized as follows. In Section 2 we give some preliminary concepts and results about the family of canonical MTL-algebras defined by a left-continuous t-norm and its residuum, also adding an involutive negation. In Section 3 we give a logical framework for $\mathcal{ALC}$-like fuzzy description logics, defining the propositional logics Λ^* and their first-order versions $\Lambda^*\forall$. In Section 3.3 the multi-modal expansions of the propositional logic Λ^* are introduced, along with their semantics. In Section 4 we define a family of attributive description languages adapted to the behavior of the logical connectives arising from a left-continuous t-norm and its residuum, and we define their knowledge bases. In Section 5 we study the translation from the fuzzy description logic language to the multi-modal language and the opposite translation. In Section 6 we study the possibility of translating reasoning tasks of FDL into logical problems of many-valued multi-modal fuzzy logics. In Section 7 we study the possibility of transferring computational results between both logical systems. We end with a section of conclusions.