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Involutive basic substructural core fuzzy logics: Involutive mianorm-based logics ☆

Eunsuk Yang

Department of Philosophy & Institute of Critical Thinking and Writing, Chonbuk National University, Jeonju, Republic of Korea

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Abstract

This paper deals with the standard completeness of *involutive* non-associative, non-commutative, substructural fuzzy logics and their axiomatic extensions. First, fuzzy systems based on involutively residuated mianorms (binary monotonic identity aggregation operations on the real unit interval [0, 1]), their corresponding algebraic structures, and their algebraic completeness results are discussed. Next, completeness with respect to algebras whose lattice reduct is [0, 1], known as *standard completeness*, is established for these systems via a construction in the style of Jenei–Montagna. These standard completeness results resolve a problem left open by Cintula, Horčík, and Noguera in the recent Handbook of Mathematical Fuzzy Logic and Review of Symbolic Logic. Finally, we briefly consider the similarities and differences between constructions of the author and Wang's Jenei–Montagna-style.

Keywords: (Involutively residuated) Mianorms; Substructural logics; Involutive logics; Non-associative, non-commutative fuzzy logics; Involutive mianorm-based logics

1. Introduction

The aim of this paper is to introduce standard completeness results for *involutive* non-associative, non-commutative, substructural fuzzy logics. Recall first some recent historical facts associated with such logics. Fuzzy logics are *substructural logics*, i.e., logics that lack various structural rules such as weakening and contraction (see [10,17]). The system **FL** (Full Lambek logic) is a prominent example of a substructural logic. This logic system does not assume the structural rules of exchange, weakening, and contraction, but instead stipulates associativity. Substructural logics that further eliminate associativity have been introduced. Galatos (and Ono) [9,11,12] introduced **GL**, a non-associative generalization of **FL** whose algebraic semantics is given by the variety of residuated lattice-ordered unital groupoids (briefly, *rlu*-groupoids) in the sense of [10].

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E-mail address: eunsyang@jbnu.ac.kr.

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According to Cintula (and Běhounek) [1,3], a (weakly implicative) logic **L** is said to be *fuzzy* if it is complete with respect to (w.r.t.) linearly ordered matrices (or algebras) and *core fuzzy* if it is complete w.r.t. *standard* algebras (i.e., algebras on the real unit interval [0, 1]). The substructural logic systems **FL** and **GL** are not (core) fuzzy logics because they are not complete w.r.t. such algebras. Their corresponding core fuzzy systems have been recently introduced. The systems **UL** (Uninorm logic) and **MIAL** (Mianorm logic, $= \mathrm{SL}^{\ell}$) are the weakest (core) fuzzy logics extending **FL** and SL, a bounded version of **GL**, respectively. In particular, **MIAL** has been introduced very recently by Cintula, Horčík, and Noguera as the *weakest* possible fuzzy logic in [4–6,15].

Although Cintula, Horčík, and Noguera have introduced weakening-free, non-associative, non-commutative substructural (core) fuzzy logics and their corresponding completeness properties, many problems still remain. For instance, standard completeness for such *involutive* logics is unresolved by [4,15]. In this paper, we characterize involutive core fuzzy logics extending **MIAL**. Specifically, this paper introduces **IMIAL** (Involutive mianorm logic), which is intended to cope with the tautologies of left-continuous conjunctive mianorms (binary monotonic identity aggregation operations) and their *involutive* residua, as InSL^{ℓ}, the involutive SL^{ℓ}.

This paper is organized as follows. In Section 2, we introduce the logic **IMIAL** and its non-associative, non-commutative axiomatic extensions, along with their corresponding algebraic semantics. In Section 3, we define mianorms and their involutive residua and provide some examples. In Section 4, we establish standard completeness for **IMIAL** and its axiomatic extensions using the Jenei–Montagna-style construction introduced in [8,16]. This consists of showing that countable, linearly ordered involutive algebras of a given variety can be embedded into linearly and densely ordered members of the same variety, which can in turn be embedded into involutive algebras with lattice reduct [0, 1]. These results were unresolved in [4,15].

Note that some Jenei–Montagna-style constructions for axiomatic extensions of **MIAL** have been provided: Yang has introduced such construction for the extensions **MICAL** (**MIAL** with the exchange axiom $\varphi\&\psi \to \psi\&\varphi$) and **IMICAL** (**MICAL** with the involution axiom $\neg\neg\varphi \to \varphi$) in [22]. Wang has performed similar construction for the extensions $\mathbf{C}_n\mathbf{UL}$ (**UL** with the n-potency axiom $\varphi^n \leftrightarrow \varphi^{n-1}$) and $\mathbf{C}_n\mathbf{IUL}$ ($\mathbf{C}_n\mathbf{UL}$ with the involution axiom) in [18,19]. The construction introduced in Section 4 is a generalization of Yang's. We may also investigate the standard completeness results in Section 4 using Wang's approach.

Note also that the present author considered the similarities and differences between the constructions of Yang and Wang's Jenei–Montagna-style in [22]. However, the similarities and differences between the constructions for *involutive* logics such as **IMICAL** and $\mathbf{C}_n\mathbf{IUL}$ have not yet been investigated. In Section 5, we briefly consider the similarities and differences between the *involutive* constructions of the author and Wang's Jenei–Montagna-style.

For convenience, we adopt notations and terminology similar to those in [3–7,14,17,20–24], and we assume reader familiarity with them (along with results found therein).

2. The logic IMIAL and its axiomatic extensions

The term *involutive mianorm-based logics* refers to substructural fuzzy logic systems with mianorm-based semantics, where the (strong) conjunction and implication connectives '&,' '→,' and '⇝' are interpreted as a left-continuous conjunctive mianorm and its involutively residuated pair. In this framework, the weakest logic is **IMIAL**. This logic and its axiomatic extensions (henceforth referred to as extensions) can be based on a countable propositional language with formulas *Fm*, built inductively as usual from a set of propositional variables *VAR*, binary connectives $\to, \rightsquigarrow, \&, \wedge, \vee$, and constants $\top, \bot, \overline{0}, \overline{1}$, with defined connectives:

df1. $\neg\varphi := \varphi \to \overline{0}$,
df2. $-\varphi := \varphi \rightsquigarrow \overline{0}$, and
df3. $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$.

We further define $\varphi_{\overline{1}}$ as $\varphi \wedge \overline{1}$. For the rest of this paper, we use the customary notations and terminology and the axiom systems to provide a consequence relation.

We start with the following axiomatization of **IMIAL**, the most basic involutive fuzzy logic introduced here.

Definition 1. ([4,5]) **IMIAL** consists of the following axiom schemes and rules:

$(\varphi \wedge \psi) \to \varphi$, $(\varphi \wedge \psi) \to \psi$ (∧-elimination, ∧-E)
$((\varphi \to \psi) \wedge (\varphi \to \chi)) \to (\varphi \to (\psi \wedge \chi))$ (∧-introduction, ∧-I)