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Cross products, invariants, and centralizers

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A B S T R A C T

An algebra V with a cross product $\times$ has dimension 3 or 7. In this work, we use 3-tangles to describe, and provide a basis for, the space of homomorphisms from $\mathsf{V}^{\otimes n}$ to $\mathsf{V}^{\otimes m}$ that are invariant under the action of the automorphism group $\mathsf{Aut}(\mathsf{V},\times)$ of V, which is a special orthogonal group when $\dim \mathsf{V} = 3$, and a simple algebraic group of type G_2 when $\dim \mathsf{V} = 7$. When $m = n$, this gives a graphical description of the centralizer algebra $\mathsf{End}_{\mathsf{Aut}(\mathsf{V},\times)}(\mathsf{V}^{\otimes n})$, and therefore, also a graphical realization of the $\mathsf{Aut}(\mathsf{V},\times)$-invariants in $\mathsf{V}^{\otimes 2n}$ equivalent to the First Fundamental Theorem of Invariant Theory. We show how the 3-dimensional simple Kaplansky Jordan superalgebra can be interpreted as a cross product (super)algebra and use 3-tangles to obtain a graphical description of the centralizers and invariants of the Kaplansky superalgebra relative to the action of the special orthosymplectic group.

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1. Introduction

A cross product algebra $(\mathsf{V}, \mathsf{b}, \times)$ is a finite-dimensional vector space V over a field $\mathbb{F}$ (assumed to have characteristic different from 2) with a nondegenerate symmetric bilinear form b and a bilinear multiplication $\mathsf{V} \times \mathsf{V} \to \mathsf{V}$, $(x, y) \mapsto x \times y$, that satisfies

$$\begin{aligned} &\mathsf{b}(x \times y, x) = 0, \\ &x \times x = 0, \\ &\mathsf{b}(x \times y, x \times y) = \mathsf{b}(x, x)\mathsf{b}(y, y) - \mathsf{b}(x, y)\mathsf{b}(y, x), \end{aligned}$$

for any $x, y \in \mathsf{V}$. Nonzero cross products exist only if $\dim_{\mathbb{F}} \mathsf{V} = 3$ or 7 (see [3,12]), and when $\mathbb{F}$ is the field of real numbers, the cross product is the familiar one from calculus in dimension 3 if b is positive definite. We relate cross product algebras to certain 3-tangle categories and give a graphical realization of the invariants and centralizer algebras of tensor powers of V under the action of its automorphism group $\mathsf{Aut}(\mathsf{V}, \times)$.

The 3-*tangle category* $\mathcal{T}$ has as objects the finite sets $[n] = \{1, 2, \ldots, n\}$ for $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$, where $[0] = \emptyset$. For any $n, m \in \mathbb{N}$, the morphisms $\mathrm{Mor}_{\mathcal{T}}([n], [m])$ are $\mathbb{F}$-linear combinations of 3-tangles, and they are generated through composition and disjoint union from the basic morphisms (basic 3-tangles) in (3.6) and (3.7). This gives a graphical calculus that enables us to describe the space $\mathsf{Hom}_{\mathsf{Aut}(\mathsf{V},\times)}(\mathsf{V}^{\otimes n}, \mathsf{V}^{\otimes m})$ of $\mathsf{Aut}(\mathsf{V}, \times)$-homomorphisms on tensor powers of V. When $\dim_{\mathbb{F}} \mathsf{V} = 7$, the group $\mathsf{Aut}(\mathsf{V}, \times)$ is a simple algebraic group of type G_2, and V is its natural 7-dimensional module (its smallest nontrivial irreducible module). When $\dim_{\mathbb{F}} \mathsf{V} = 3$, $\mathsf{Aut}(\mathsf{V}, \times)$ is the special orthogonal group $\mathsf{SO}(\mathsf{V}, \mathsf{b})$.

From the properties of the cross product, we construct three homomorphisms (when $\dim_{\mathbb{F}} \mathsf{V} = 7$, they are given in (4.6), (4.3), and when $\dim_{\mathbb{F}} \mathsf{V} = 3$, in (5.5), (5.2)). Applying methods similar to those in [6,7,4], we show in each case (see Theorems 4.10 and 5.13 for the precise statements) that these homomorphisms correspond to a set Γ^* consisting of three 3-tangle relations, and the following result holds. In the statement, the 3-tangles must satisfy some additional constraints. When $\dim_{\mathbb{F}} \mathsf{V} = 3$, these constraints are incorporated in the definition of "normalized" 3-tangles.

Theorem 1.1. *Let $n, m \in \mathbb{N}$ and assume that the characteristic of $\mathbb{F}$ is 0. Let $(\mathsf{V}, \mathsf{b}, \times)$ be a vector space V endowed with a nonzero cross product $x \times y$ relative to the nondegenerate symmetric bilinear form b.*

(a) *The classes modulo Γ^* of (normalized) 3-tangles $[n] \to [m]$ without crossings and without any of the subgraphs in (4.13) form a basis of $\mathrm{Mor}_{\mathcal{T}_{\Gamma^*}}([n], [m])$.*
(b) *There is a functor $\mathcal{R}_{\Gamma^*}$ giving a linear isomorphism*

$$\mathrm{Mor}_{\mathcal{T}_{\Gamma^*}}([n], [m]) \to \mathsf{Hom}_{\mathsf{Aut}(\mathsf{V},\times)}(\mathsf{V}^{\otimes n}, \mathsf{V}^{\otimes m}).$$

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