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Cartesian products of directed graphs with loops

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ABSTRACT

We show that every nontrivial finite or infinite connected directed graph with loops and at least one vertex without a loop is uniquely representable as a Cartesian or weak Cartesian product of prime graphs. For finite graphs the factorization can be computed in linear time and space.

1. Introduction

This note treats finite and infinite directed graphs with or without loops. It is shown that every connected, finite or infinite directed graph, with at least one vertex without a loop, is uniquely representable as a Cartesian or weak Cartesian product of prime graphs, i.e. graphs that cannot be represented as a product of two nontrivial graphs. Moreover, we show that the factorization can be computed in linear time and space for finite graphs.

The note extends and unifies results by Boiko et al. [1] about the Cartesian product of finite undirected graphs with loops, and Crespelle and Thierry [2] about finite directed graphs. For infinite graphs it generalizes a result by Miller [8] and Imrich [6] about the weak Cartesian product.

In 1960 the unique prime factorization property of connected finite graphs with respect to the Cartesian product was proved to hold by Sabidussi [9] and in 1963 an independent proof was published by Vizing [10]. Sabidussi also introduced the weak Cartesian product.

Sabidussi's proof is non-algorithmic. For undirected graphs the first factorization algorithm was published by Feigenbaum, Hershberger and Schäffer [4]. Its complexity is $O(n^{4.5})$, where n is the number of vertices of the graph. Subsequently the complexity was reduced by a number of authors. The latest improvement, Imrich and Peterin [7], is linear with respect to the number of edges.

The first factorization algorithm for directed graphs was published by Feigenbaum [3]. It starts with the undirected decomposition provided by [4]. Crespelle and Thierry [2] also use an undirected decomposition and then compute the prime factorization of the directed graph in linear time. In Section 3 we present a simpler algorithm of the same complexity and extend it to the case when loops are allowed.

Infinite graphs are treated in Sections 4 and 5. In Section 4 we consider the weak Cartesian product, which is a connected component of the Cartesian product of infinitely many connected non-trivial factors, and provide a unified, short proof for the associativity of all types of Cartesian products that are considered in this note. In Section 5 the paper is completed by a direct proof of the unique factorization property of finite or infinite directed graphs with loops with respect to the Cartesian or the weak Cartesian product. This result not only extends the one in [8] and [6] for simple graphs, see also [5], but the proof is much shorter and more transparent.

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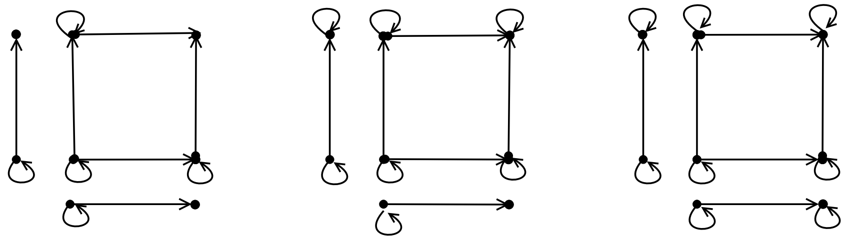

Fig. 1. Cartesian products of graphs with loops.

2. Preliminaries

A directed graph G with loops consists of a set $V(G)$ of vertices together with a subset $A(G)$ of $V(G) \times V(G)$. The elements of $A(G)$ are called *arcs* and are ordered pairs of vertices. If ab is an arc, we call a its *origin*, b its *terminus*, and also refer to a and b as *endpoints*. The vertex b is also called an *out-neighbor* of a, and a an *in-neighbor* of b.

We allow that a equals b. In this case we speak of a *loop* at vertex a and say the vertex a is *looped*. Notice that it is possible that $A(G)$ contains both ab and ba.

For better readability we often write $v \in G$ instead of $v \in V(G)$, and $e \in G$ instead of $e \in A(G)$ or $E(G)$.

$\overrightarrow{\Gamma_0}$ denotes the class of directed graphs with loops. To every directed graph $G \in \overrightarrow{\Gamma_0}$ we also define its *shadow* $\mathcal{S}(G)$. It has the same vertex set as G and its set of edges $E(G)$ consists of all unordered pairs $\{a, b\}$ of distinct vertices for which ab, ba or both are in $A(G)$. To indicate that $\{a, b\}$ is an edge, we will again use the notation ab. We say that G is *weakly connected* if $\mathcal{S}(G)$ is connected. When it is clear from the context we will simply write connected instead of weakly connected. The *distance* $d_G(u, v)$ between two vertices equals to $d_{\mathcal{S}(G)}(u, v)$.

Similarly, we set the *degree* $d_G(u)$ of a vertex equal to $d_{\mathcal{S}(G)}(u)$ and let δ denote the minimum degree of G.

Furthermore, we write Γ for the class of simple graphs, Γ_0 for the class of simple graphs with loops, and $\overrightarrow{\Gamma}$ for the class of directed graphs without loops. Clearly Γ, Γ_0 and $\overrightarrow{\Gamma}$ are embeddable into $\overrightarrow{\Gamma_0}$ if one replaces edges of graphs in Γ or Γ_0 by pairs of arcs with opposite directions and considers the fact that graphs in Γ and $\overrightarrow{\Gamma}$ have no loops.

The *Cartesian product* $G \,\square\, H$ *of graphs in* $\overrightarrow{\Gamma_0}$ is defined on the Cartesian product $V(G) \times V(H)$ of the vertex sets of the factors. Its set of arcs is

$$A(G \,\square\, H) = \left\{(x, u)(y, v) \mid xy \in A(G) \text{ and } u = v, \text{ or}, x = y \text{ and } uv \in A(H)\right\}.$$

If G and H have no loops, then this is also the case for $G \,\square\, H$. To obtain the definition of the Cartesian product of undirected graphs, one just replaces $A(G)$ by $E(G)$. Hence, the new definition generalizes the definition of the Cartesian product of simple graphs, directed graphs, and simple graphs with loops. Note that $\mathcal{S}(G \,\square\, H) = \mathcal{S}(G) \,\square\, \mathcal{S}(H)$.

Similarly, if G is a graph in Γ_0 or $\overrightarrow{\Gamma_0}$ and $\mathcal{N}(G)$ denotes the graph obtained from G by removal of the loops, then $\mathcal{N}(G \,\square\, H) = \mathcal{N}(G) \,\square\, \mathcal{N}(H)$.

Clearly Cartesian multiplication is commutative and the trivial graph K_1 is a unit, i.e. $G \square K_1 \cong G$ for every graph G. It is well known that it is associative in Γ. That Cartesian multiplication is associative for finite graphs in $\overrightarrow{\Gamma}$ and Γ_0 was shown in [3], resp. [1]. For $\overrightarrow{\Gamma_0}$ we present a proof in Section 4.

A nontrivial, connected directed graph G with at least one unlooped vertex is called *irreducible* or *prime* with respect to Cartesian multiplication if, for every factorization $G = A \,\square\, B$, either A or B has only one vertex. It is known that prime factorization for finite connected graphs in Γ, $\overrightarrow{\Gamma}$ and Γ_0, where one has to require at least one unlooped vertex, is unique up to the order and isomorphisms of the factors. This was shown, respectively, in [3,9,10], and [1]. Furthermore, by [6,8], infinite connected graphs in Γ have the unique factorization property with respect to the weak Cartesian product. Here we show that unique factorization also holds for connected finite and infinite graphs in $\overrightarrow{\Gamma_0}$, if there exists at least one unlooped vertex, with respect to the Cartesian or the weak Cartesian product, see Section 5.

For our proofs and algorithms we need projections and layers. The ith projection $p_i : V(G) \to V(G_i)$ of a product $\prod_{i=1}^{k} G_i$ is defined by $(v_1, \ldots, v_k) \mapsto v_i$, and the G_i-*layer* G_i^v *through a vertex* $v = (v_1, \ldots, v_k) \in G$ is the subgraph induced by the set

$$\{(w_1, \ldots, w_k) \in V(G) \mid w_j = v_j \text{ for all } j \neq i, 1 \leq j \leq k\}.$$

Sometimes we will also use the notation p_{G_i} instead of p_i.

For unlooped graphs the restriction of projection p_i to G_i^v is isomorphic to G_i, but this does not hold for graphs with loops unless G_i^v contains an unlooped vertex, see the left part of Fig. 1. The other part of the figure shows that directed graphs with no unlooped vertex need not have unique prime factorizations.

Usually we color the edges of a product $\prod_{i=1}^{k} G_i$ by k colors such that the edges of the G_i-layers are assigned color i, and denote the color of uv by $c(uv)$.

We need the next lemma about metric properties of a product.