Ample continua in Cartesian products of continua

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ABSTRACT

We show that the Cartesian product of the arc and a solenoid has the fupcon property, therefore answering a question raised by Illanes. This combined with Illanes’ result implies that the product of a Knaster continuum and a solenoid has the fupcon property, therefore answering a question raised by Bellamy and Łysko in the affirmative. Finally, we show that a product of two Smith’s nonmetric pseudo-arcs has the fupcon property.

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1. Introduction

The present paper is concerned with the property of having arbitrarily small open neighborhoods for continua in Cartesian products of continua; i.e. given a continuum $M \subseteq X \times Y$ we are interested if

(*) for every open neighborhood $U$ of $M$ there exists an open and connected set $V$ such that $M \subseteq V \subseteq U$.

The property (*) is closely related to the property of being an ample\(^1\) continuum in the product. Recall that $M$ is ample in $X \times Y$ provided that for each open subset $U \subseteq X \times Y$ such that $M \subseteq U$, there exists a subcontinuum $L$ of $X \times Y$ such that $M \subseteq \text{int}_{X \times Y}(L) \subseteq L \subseteq U$. In fact, according to [1], the two

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\(^1\) The notion of an ample continuum was introduced by Prajs and Whittington in [10].

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properties are equivalent in the class of Kelley continua. Motivation for the study of ample continua comes from fact that in the hyperspace $C(X \times Y)$ of subcontinua of $X \times Y$ ample continua are the points where $C(X \times Y)$ is locally connected. In this context in [1] Bellamy and Łysko studied the $fupcon$ property of Cartesian products. The product of continua $X \times Y$ has the $fupcon$ property if whenever $M \subseteq X \times Y$ is a continuum with full projections onto coordinate spaces (i.e. $\pi_X(M) = X$ and $\pi_Y(M) = Y$) then $M$ has the property (*), and the notion naturally generalizes to Cartesian products of more than two continua. Bellamy and Łysko showed that arbitrary Cartesian products of Knaster continua and arbitrary Cartesian products of pseudo-arcs have the $fupcon$ property. Furthermore, the property (*) for subcontinua of such products is in fact equivalent to the property of having full projections onto all coordinate spaces. The authors also showed that the diagonal in a Cartesian square $G$ of a compact and connected topological group has the property (*) if and only if $G$ is locally connected, and therefore if $G$ is a solenoid then $G \times G$ does not have the $fupcon$ property. Important related results on ample diagonals can be found in the recent work of Prajs [9]. Motivated by the aforementioned results, Bellamy and Łysko raised the following question.

**Question 1** (Bellamy & Łysko [1]). Let $K$ be a Knaster continuum and $S$ be a solenoid. Does $K \times S$ have the $fupcon$ property?

A partial step towards a solution to the above problem was achieved by Illanes, who showed the following.

**Theorem A** (Illanes [7]). Let $X$ be a continuum such that $X \times [0, 1]$ has the $fupcon$ property. Then for each Knaster continuum $K$, $X \times K$ has the $fupcon$ property.

Consequently, Question 1 was reduced to the following, potentially simpler problem.

**Question 2** (Illanes [7]). Let $S$ be a solenoid. Does $[0, 1] \times S$ have the $fupcon$ property?

We answer this question in the affirmative, and in turn obtain positive answer to Question 1.

**Theorem 1.1.** Let $S$ be a solenoid. Then $[0, 1] \times S$ has the $fupcon$ property.

**Theorem 1.2.** Let $S$ be a solenoid and $K$ be a Knaster continuum. Then $K \times S$ has the $fupcon$ property.

In 1985 M. Smith [11] constructed a nonmetric pseudo-arc $M$; i.e. a Hausdorff chainable, homogeneous, hereditary equivalent and hereditary indecomposable continuum. This continuum has been recently used by the first and third author to provide a new counterexample to Wood’s Conjecture in the isometric theory of Banach spaces [2]. Relying on the result of Bellamy and Łysko that products of metric pseudo-arcs have the $fupcon$ property, we shall show that their result holds also for products of $M$.

**Theorem 1.3.** Let $M$ be Smith’s nonmetric pseudo-arc. Any Cartesian power of $M$ has the $fupcon$ property.

Earlier, Lewis showed [8] that for any 1-dimensional continuum $X$ there exists a 1-dimensional continuum $X_P$ that admits a continuous decomposition into pseudo-arcs, and whose decomposition space is homeomorphic to $X$. Recently, Boroński and Smith [3] extended Lewis’ result to continuous curves of Smith’s nonmetric pseudo-arc. In particular, given any metric 1-dimensional continuum $X$ there exists a continuum $X_M$ that admits a continuous decomposition into nonmetric pseudo-arcs, and whose decomposition space is homeomorphic to $X$. $X_M$ can be seen as “$X$ of nonmetric pseudoarcs”. Here we observe that using the method of proof of Theorem 1.3 one obtains the following generalization.

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2 The abbreviation $fupcon$ stands for **full projections imply connected open neighborhoods.** It was introduced by Illanes in [7].
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