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# Optimal control of the customer dynamics based on marketing policy

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ABSTRACT

We consider an optimal control problem for a non-autonomous model of ODEs that describes the evolution of the number of customers in some firm. Namely we study the best marketing strategy. Considering a $L^2$ cost functional, we establish the existence and uniqueness of optimal solutions, using an inductive argument to obtain uniqueness on the whole interval from local uniqueness. We also present some simulation results, based on our model, and compare them with results we obtain for an $L^1$ cost functional. For the $L^1$ cost functional the optimal solutions are of bang-bang type and thus easier to implement, because at every moment possible actions are chosen from a finite set of possibilities. For the autonomous case of $L^2$ problem, we show the effectiveness of the optimal control strategy against other formulations of the problem with simpler controls.



## 1. Introduction

Firms spend millions of euros on marketing budgets. The CMO report conducted in 2017 by the Fuqua School of Business, the American Marketing Association and Delloite shows that firms allocate, in general, between 10% and 20% of their revenues on marketing budgets, depending on the sector where they operate. Considering the high amounts involved, it is very important to optimize that allocation. However, as stated by Gupta and Steenburgh [6] allocating marketing resources is a complex decision that until recently has been done based on very simple heuristics or decision rules.

Among marketing decisions and strategy is the decision to invest in referrals programs. These programs encourages current customers to recruit new customers based on rewards [15]. Contrary to other marketing programs purely based on spontaneous word-of-mouth, referral programs are marketer directed with possibility to control message content [3]. However, studies that help marketers to decide about the resource allocation to referral programs are scarce.

For decades, firms have been searching for the best way to maximize profits and reduce costs. Classical models usually look for ways that help firms allocating their marketing resources while maximizing profits [1]. However, more recently models have tried to maximize customer equity (the net present value of the future profit flow over a customers lifetime [12] through an optimal marketing resources allocation [8]). In this sense, and based on the assumption that the number of customers in a market is limited, it is important to attract/capture new customers at earliest as possible as, otherwise,

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they can be attracted/captured by competitors. At the same time, a customer late attraction/caption will also reduce their customer equity.

Following the growing interest of social networks by product marketing managers, recently the classic epidemiologic models have been applied with success to specific marketing communication strategy, commonly referred as viral marketing. An application of epidemiology to a real-world problem can be found in [14].

Previously, in [17] the authors of this work proposed a compartmental model suitable to describe the dynamics of the number of customers of a given firm. That model was given by a system of ordinary differential equations whose variables correspond to groups of customers and potential customers divided according to their profile and whose parameters reflect the structure of the underlying social network and the marketing policy of the firm. Understand the flows between these groups and its consequences on the raise of customers of the firm was the main goal. Highlight the usefulness of these models in helping firms deciding their marketing policy was another objective.

Election campaign managers and companies marketing products/services managers, are interested in spreading a message by a given deadline, using limited resources. So, the optimal resource allocation over the time of the campaign is required and the formulation of such situation as an optimal control problem is suggested. In [7], that problem is tackled using two epidemic models, a SIS and a SIR.

In this paper, we consider a modified version of model [17], governed by the system of ordinary differential equations:

$$\begin{cases} \dot{R} = -\lambda_2 R + \lambda_1 C - \gamma(t)R + \alpha\,\beta(t)PR/N \\ \dot{C} = -\lambda_1 C + \lambda_2 R - \gamma(t)C + (1-\alpha)\beta(t)PR/N \\ \dot{P} = -\beta(t)PR/N + \gamma(t)(R+C) \end{cases} \tag{1}$$

with initial conditions

$$R(0), C(0), P(0) \geqslant 0,$$

where $R$ is the number of referral customers, $C$ is the number of regular customer, $P$ is the number of potential customers and $N = R + C + P$.

The parameters of the model represent the following: $\lambda_1$ is the natural transition rate between regular customers and referral costumers, given by the number of regular customers that become referral customers without external influence over the number of regular customers (by “without external influence” we mean without being influenced by referral customers); $\lambda_2$ is the natural transition rate between referral costumers and regular customers, given by the number of referral customers that become regular customers without external influence over the number of referral customers; $\gamma(t)$ is the time varying customer defection rate, equal to the number of customers that cease to be customers over the number of customers (we assume that this rate is the same among regular and referral costumers); $\beta(t)$ is the pull effect due to marketing campaigns, corresponding to the quotient of the outcome of marketing campaigns by the number of potential customers (by “outcome of marketing campaigns” it is meant the number of potential customers that become customers in the sequence of marketing campaigns per unitary marketing cost per time unit); finally, $\alpha$ is the percentage of referral costumers among the new customers.

The main difference between the above model and the model presented in [17] is that, instead of using a single compartment corresponding to potential clients and assuming that a fixed percentage of those potential clients are referral clients, in [17], the potential clients are divided in two subpopulations (corresponding to potential regular clients and potential referral clients).

We stress that by using time varying parameters, $\beta(t)$ and $\gamma(t)$, in (1) we obtain a non-autonomous model that is potentially more realistic. The objective of this paper is to consider an optimal control problem for such non-autonomous model.

## 2. Optimal control problem

Inspired in [7], we assume that the campaigner can allocate its resources in two ways. At time $t$, he can directly recruit individuals from the population with rate $u_1(t)$, to be clients (via publicity in mass media). In addition, he can incentivize clients to make further recruitments (e.g. monetary benefits, discounts or coupons to current customers who refer their friends to buy services/products from the company). This effectively increases the “spreading rate” at time $t$ from $\beta(t)$ to $\beta(t) + u_2(t)$ where $u_2(t)$ denotes the “word-of-mouth” control signal which the campaigner can adjust at time $t$.

The diagram of the non-autonomous model we propose is shown in Fig. 1. The respective equations are the following:

$$\begin{cases} \dot{R} = -\lambda_2 R + \lambda_1 C - \gamma(t)R + \alpha_1\, u_1 P + \alpha_2\,(\beta(t) + u_2)PR/N \\ \dot{C} = -\lambda_1 C + \lambda_2 R - \gamma(t)C + (1-\alpha_2)(\beta(t) + u_2)PR/N + u_1(1-\alpha_1)P \\ \dot{P} = -(\beta(t) + u_2)PR/N - u_1 P + \gamma(t)R + \gamma(t)C \end{cases} \tag{2}$$

with initial conditions

$$R(0), C(0), P(0) \geqslant 0. \tag{3}$$

The parameters $u_1$, $u_2$ will be taken in the space $L^\infty$ functions such that $u_1 \in [0, u_{1\max}]$ and $u_2 \in [0, u_{2\max}]$.

Our purpose is to minimize the number of potential customers and the cost associated to the control of the marketing campaigns. To obtain the best reduction in the number of potential customers, we minimize the evolution history,