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Current-flow efficiency of networks

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HIGHLIGHTS

- Current-flow efficiency metric considers multipath effect of many real networks.
- Current-flow efficiency metric can handle the disconnected network.
- Analyze how the network topological structure affects the current-flow efficiency.
- Use this metric in measuring network efficiency and finding vital nodes or edges.

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ABSTRACT

Many real-world networks, from infrastructure networks to social and communication networks, can be formulated as flow networks. How to realistically measure the transport efficiency of these networks is of fundamental importance. The shortest-path-based efficiency measurement has limitations, as it assumes that flow travels only along those shortest paths. Here, we propose a new metric named current-flow efficiency, in which we calculate the average reciprocal effective resistance between all pairs of nodes in the network. This metric takes the multipath effect into consideration and is more suitable for measuring the efficiency of many real-world flow equilibrium networks. Moreover, this metric can handle a disconnected graph and can thus be used to identify critical nodes and edges from the efficiency-loss perspective. We further analyze how the topological structure affects the current-flow efficiency of networks based on some model and real-world networks. Our results enable a better understanding of flow networks and shed light on the design and improvement of such networks with higher transport efficiency.

1. Introduction

Many complex networks exist in the form of flow networks [1]. For instance, people or goods flow through transportation networks, electric current flows through electrical transmission networks, information flows through communication networks, epidemics and opinions flow through social networks. Assessing how cost-efficient flow is exchanged over such networks is of fundamental importance in network analysis. A well-known metric for this purpose is the shortest-path efficiency [2], defined as the average inverse shortest-path length under the assumption that flows move along the shortest possible path across the network. This measurement can be calculated for both connected and disconnected networks and has been applied in many different fields [3–5], especially in assessing efficiency degradation when a network is damaged [6,7]. However, for many networks, flow does not follow only the shortest path — for example, rumors propagate

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along many possible paths established based on personal relationships among individuals in a social network [8], traffic flow between the same origin–destination pair is distributed along multiple routes in a road network [9], and electric current or fluid flows in a power grid or pipeline network according to the specific physical law. For such networks with multipath transport behavior, the shortest-path efficiency cannot fully reflect the efficiency of the system.

To rationally measure the multipath efficiency of general flow networks, several alternative graph metrics that take into account the multipath transport effect have been proposed [10–16]. One such metric is network communicability [10,11], which quantifies the amount of information that can flow from one node to another in a network. The communicability between two nodes in a network is the weighted sum of all possible paths starting from one node to another, in which the longer paths are assigned smaller weights. Based on the concept of network communicability, the average communicability distance [12] and communicability angle [13] have been proposed to measure the communication efficiency of networks. These metrics are of great interest as contributions from all possible paths are considered; however, network efficiency is evaluated from the perspective of maximum flow, while the time or effort required to transport these flows is neglected.

Another interesting metric for network efficiency is the effective graph resistance [15,16], which is defined as the accumulated effective resistance between all pairs of nodes in a network. Many dynamical processes in highly different systems, including random walks in a social network [8], congested traffic flow in a road network [9,17], viscous flow in a pipe network [18] and force equilibrium in a mechanical network [19], are mathematically equivalent to Kirchhoff's equilibrium equation for an electrical circuit [19,20]; thus, the effective graph resistance is suitable to measure the efficiency of these flow equilibrium networks. However, this metric cannot be applied to a disconnected network because the effective resistance of a disconnected graph is always infinite [15]. This shortcoming causes the effective graph resistance to be unsuitable for assessing the efficiency loss caused by the removal of nodes or edges, leading to separation of the original network into disconnected clusters.

In this paper, we propose a new metric named current-flow efficiency, which is defined as the average reciprocal effective resistance between all pairs of nodes in a network. This metric takes the multipath effect into consideration, which allows for a more realistic measure of efficiency for many real-world flow networks. More importantly, it can handle a disconnected graph, in which the conductance between two disconnected nodes is zero and the graph's current-flow efficiency is between zero and one. This metric can be used to identify critical nodes and edges from the efficiency-loss perspective.

2. Definition of current-flow efficiency

We view an undirected weighted graph as an electrical network with only pure resistors and ideal voltage sources, in which each node $i \in \{1, 2, \ldots, N\}$ is possibly connected to a voltage source and each edge is a resistor with resistance r_{ij} (in this paper we just consider unweighted graph and set $r_{ij} = 1$ for all edges). The application of Ohm's law to each edge (i, j) yields the current flowing from i to j

$$I_{ij} = \frac{v_i - v_j}{r_{ij}} = a_{ij}(v_i - v_j), \tag{1}$$

where v_i is the voltage at node i, a_{ij} is the conductance (inverse resistance) and $a_{ij} = 0$ if two nodes are not connected by a resistor. Let one unit of current be injected into the network at a source node s and extracted at a target node t. Kirchhoff's current law states that the total current flow into or out of any node is zero, which implies that the voltages satisfy the nodal equilibrium equation [8,16]

$$\sum_j a_{ij}(v_i - v_j) = \begin{cases} +1; & \text{for } i = s, \\ -1; & \text{for } i = t, \\ 0; & \text{otherwise,} \end{cases} \tag{2}$$

or the matrix form

$$(D - A)V = I, \tag{3}$$

where D is the diagonal matrix with elements $D_{ii} = \sum_j a_{ij}$ and the vector I has elements

$$I_i = \begin{cases} +1; & \text{for } i = s, \\ -1; & \text{for } i = t, \\ 0; & \text{otherwise.} \end{cases} \tag{4}$$

Notice that $L = D - A$ is the graph Laplacian matrix. If L^+ is the Moore–Penrose pseudoinverse of L, then the solution of Eq. (3) is

$$V = L^+ I. \tag{5}$$

The effective resistance [15,16] (also called resistance distance) R_{ij} between i and j is the potential difference of nodes i and j when a unit of current is injected at the source i and removed at the target j:

$$R_{ij} = v_i - v_j = (L^+_{ii} - L^+_{ij}) - (L^+_{ji} - L^+_{jj}) = L^+_{ii} + L^+_{jj} - 2L^+_{ij}. \tag{6}$$