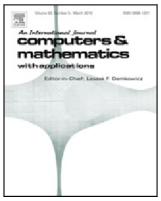
Computers and Mathematics with Applications 74 (2017) 3111–3128

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa



Numerical resolution of an electromagnetic inverse medium problem at fixed frequency

M. de Buhan [a], M. Darbas [b,*]

[a] CNRS, UMR 8145, MAP5, Université Paris Descartes, Sorbonne Paris Cité, France
[b] CNRS UMR 7352, LAMFA, Université de Picardie Jules Verne, Amiens, France

ARTICLE INFO

Article history:
Received 26 September 2016
Received in revised form 28 July 2017
Accepted 2 August 2017
Available online 30 August 2017

Keywords:
Inverse medium problem
Maxwell's equations
Cost functional
Minimization iterative process
Adaptive eigenspace method
Numerical reconstruction

ABSTRACT

The aim of this paper is to solve numerically the inverse problem of determining the complex refractive index of an electromagnetic medium from partial boundary field measurements at a fixed frequency. The governing equations are the time-harmonic Maxwell equations formulated in electric field in a two-dimensional bounded domain. We express the inverse problem as the minimization of a cost function representing the difference between the measured and predicted fields. Our numerical reconstruction algorithm combines the BFGS method and an iterative process, called the Adaptive Eigenspace Inversion. The unknown complex coefficient is expanded in terms of eigenfunctions of an elliptic operator. Both the eigenspace and the mesh are iteratively adapted during the minimization procedure. Numerical experiments illustrate the performance of the reconstruction for various configurations.

1. Introduction

The present paper deals with the numerical resolution of an electromagnetic inverse medium problem. More precisely, we consider the problem of determining the complex refractive index of a medium, namely the dielectric permittivity (real part) and the electric conductivity (imaginary part), from a finite number of boundary field measurements at a fixed frequency. The governing equations are the time-harmonic Maxwell equations formulated in electric field in a two-dimensional bounded domain. Such an electromagnetic inverse problem arises in various areas of science and engineering with many applications, e.g. in medical imaging, geophysical exploration or non-destructive testing. For instance, microwave imaging (electromagnetic high frequencies) is under investigation for cancer screening or brain stroke detection (see Tournier et al. [1,2]). Numerical methods that are able to highlight dielectric contrast between normal and possibly abnormal tissue are of interest.

From a mathematical point of view, the considered inverse medium problem is severely ill-posed and we refer the reader to the book [3] by Romanov and Kabanikhin. Indeed, coefficients of elliptic problems (like the time-harmonic Maxwell problem) in a bounded domain are uniquely determined by the entire Dirichlet-to-Neumann map on the whole boundary of the domain (e.g. Ola, Päivärinta and Somersalo [4], Caro, Ola and Salo [5], Kenig, Salo and Uhlmann [6] and references therein). A typical problem of this type is Calderón's inverse conductivity problem [7]. Nevertheless, it is legitimate to search for reconstruction methods using partial information on the Dirichlet-to-Neumann map, which is often the case in practice. Several analytical and numerical studies have been devoted to the detection of inhomogeneities in the electromagnetic parameters of a body. Ammari et al. (e.g. [8,9]) have introduced asymptotic methods to reconstruct small amplitude

---

[*] Corresponding author.
E-mail address: marion.darbas@u-picardie.fr (M. Darbas).

http://dx.doi.org/10.1016/j.camwa.2017.08.002
0898-1221/© 2017 Elsevier Ltd. All rights reserved.

perturbations in coefficients from measurements on a part of the boundary. This yields constructive numerical methods for the localization of electromagnetic defects (e.g. Ammari et al. [10], Asch and Mefire [11], Darbas and Lohrengel [12]). Concerning minimization approaches, Beilina et al. (e.g. [13,14]) have developed an adaptive finite element method based on a posteriori estimates. Other successful methods have been proposed in the literature for the numerical solution of the electromagnetic scattering medium problem. Data are in this case measurements of the far-field pattern of the scattered field. Without being exhaustive, we can mention among them the linear sampling method of Haddar and Monk [15], a preconditioned Newton method initiated by Hohage [16], or a regularized recursive linearization method used by Bao and Li [17].

Here, we propose to formulate the inverse medium problem as the minimization of a cost function representing the difference between the measured and predicted fields. To solve the minimization problem, we use a gradient-based quasi-Newton algorithm. The main goal of this paper is to present a reconstruction method for the unknown complex refractive index of the medium from boundary measurements. The idea is to consider the space spanned by some eigenvectors of the Laplacian operator as the approximation space for the unknown coefficient. Then, the method uses an iterative process to adapt the mesh and the basis of eigenfunctions to the previous approximation during the minimization procedure. This method is called *Adaptive Eigenspace Inversion*. We compare it with a more standard choice given by a linear piecewise approximation of the coefficient. The Adaptive Eigenspace Inversion (AEI or referred sometimes as AI) method has been initially proposed for the viscoelastic system by de Buhan and Osses [18]. It has been successfully applied to an inverse scattering problem for the wave equation in a paper of de Buhan and Kray [19]. In both cases, time evolution problems of hyperbolic type are treated. The geometric optics condition of Bardos–Lebeau–Rauch [20], which allows that the associated inverse problems are uniquely solved, is satisfied. More precisely, the part of the boundary where the measurements (namely the normal derivative of the solution) are recorded, and the final observation time control geometrically the domain in the sense of [20]. The application of the AEI method to time-harmonic problems is a new area of research. This is the aim of the present work in electromagnetics, and also the one of Grote, Kray and Nahum which study the resolution of an inverse problem for the Helmholtz equation. In [21,22], they proposed to combine a new AEI method and a frequency stepping process where the frequency of the incident field is iteratively increasing, with successful results. In the inverse problem we consider in this paper, we restrict ourselves to a fixed frequency. This is motivated by biomedical applications that we have in mind [1,2]. Biological tissues are dispersive [23] that is to say their dielectric properties are frequency-dependent. We are not interested in finding this dependency law but only in discriminating between healthy and abnormal tissues. This can be achieved with a single frequency and changing the frequency does not provide more information.

The remainder of the paper is organized as follows. In Section 2, we present the forward problem under consideration. The formulation of the inverse problem is addressed in Section 3. It is formulated as a nonlinear optimization problem. The key-point is the evaluation of the gradient of the cost function. We propose to use the adjoint method. In Section 4, we describe the reconstruction method based on the AEI method from a methodological point of view. Then, in Section 5, various numerical results are reported to discuss the advantages and limits of the method, with a particular interest in discontinuous coefficients. Finally, we give some concluding remarks.

2. The forward problem

Let Ω be a bounded domain in $\mathbb{R}^2$ with a smooth boundary $\Gamma := \partial\Omega$. We denote by μ_0 and ε_0 the permeability and the permittivity of the vacuum. We assume that Ω is filled with a non-magnetic (i.e. constant permeability $\mu = \mu_0$) and isotropic medium of dielectric permittivity $\varepsilon = \varepsilon(\mathbf{x})$ and electrical conductivity $\sigma = \sigma(\mathbf{x})$, $\mathbf{x} \in \Omega$. We consider the system of 2D Maxwell equations

$$\nabla \times \boldsymbol{\mathcal{E}} = -\partial_t \mathcal{B}, \qquad \nabla \times \mathcal{H} = \partial_t \boldsymbol{\mathcal{D}} + \boldsymbol{\mathcal{J}}, \text{ in } \Omega, \tag{2.1}$$

where $(\boldsymbol{\mathcal{E}}, \mathcal{H})$ are the electric and magnetic fields, $(\mathcal{B}, \boldsymbol{\mathcal{D}})$ are the magnetic and electric flux densities and $\boldsymbol{\mathcal{J}}$ represents the electrical current density. Notice that in two dimensions, the vector rotational operator is defined for a scalar function φ by $\boldsymbol{\nabla} \times \varphi = (\partial_2\varphi, -\partial_1\varphi)^t$, whereas the scalar rotational operator acting on a vector field $\mathbf{v} = (v_1, v_2)$ is given by $\nabla \times \mathbf{v} = \partial_1 v_2 - \partial_2 v_1$. We assume linear and isotropic constitutive relations

$$\mathcal{B} = \mu_0 \mathcal{H}, \quad \boldsymbol{\mathcal{D}} = \varepsilon \boldsymbol{\mathcal{E}}, \text{ and } \boldsymbol{\mathcal{J}} = \sigma \boldsymbol{\mathcal{E}}. \tag{2.2}$$

The wave equation for the electric field with no source term can be derived from (2.1) and (2.2) by eliminating the magnetic field as

$$\boldsymbol{\nabla} \times (\nabla \times \boldsymbol{\mathcal{E}}) + \mu_0(\varepsilon \partial_t^2 \boldsymbol{\mathcal{E}} + \sigma \partial_t \boldsymbol{\mathcal{E}}) = 0.$$

Considering the harmonic dependence in time of the form $\boldsymbol{\mathcal{E}}(t, \mathbf{x}) = \Re(e^{-i\omega t}\mathbf{E}(\mathbf{x}))$, the electric field $\mathbf{E}$ satisfies the following equation in the frequency domain

$$\boldsymbol{\nabla} \times (\nabla \times \mathbf{E}) - k^2 \kappa \mathbf{E} = \mathbf{0}, \text{ in } \Omega, \tag{2.3}$$

where $k = \omega\sqrt{\varepsilon_0\mu_0}$ is the wavenumber and the function

$$\kappa(\mathbf{x}) = \frac{1}{\varepsilon_0}\left(\varepsilon(\mathbf{x}) + i\frac{\sigma(\mathbf{x})}{\omega}\right), \ \mathbf{x} \in \Omega, \tag{2.4}$$