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Robust combinatorial optimization with knapsack uncertainty

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ABSTRACT

We study in this paper min max robust combinatorial optimization problems for an uncertainty polytope that is defined by knapsack constraints, either in the space of the optimization variables or in an extended space. We provide exact and approximation algorithms that extend the iterative algorithms proposed by Bertsimas and Sim (2003). We also study the limitation of the approach and point out $\mathcal{NP}$-hard situations. Then, we approximate axis-parallel ellipsoids with knapsack constraints and provide an approximation scheme for the corresponding robust problem. The approximation scheme is also adapted to handle the intersection of an axis-parallel ellipsoid and a box.

1. Introduction

Robust optimization pioneered by [1] has become a key framework to address the uncertainty that arises in optimization problems. Stated simply, robust optimization characterizes the uncertainty over unknown parameters by providing a set that contains the possible values for the uncertain parameters and considers the worst-case over the set. The popularity of robust optimization is largely due to its tractability for uncertainty handling, since linear robust optimization problems are essentially as easy as their deterministic counterparts for many types of convex uncertainty sets [1], contrasting with the well-known difficulty of stochastic optimization approaches. In addition, robust optimization offers conservative approximation to stochastic programs with probabilistic constraints by choosing appropriate uncertainty sets [2–4].

The picture is more complex when it comes to robust combinatorial optimization problems. Let N denote a set of indices, with $|N| = n$, and $\mathcal{X} \subset \{0,1\}^n$ be the feasibility set of a combinatorial optimization problem, denoted by CO. Given a bounded uncertainty set $\mathcal{U} \subset \mathbb{R}^n_+$, we consider in this paper the min max robust counterpart of CO, defined as

$$CO(\mathcal{U}) \qquad \min_{x\in\mathcal{X}} \max_{\xi\in\mathcal{U}} \xi^T x. \tag{1}$$

It is well-known (e.g. [5,6]) that a general uncertainty set $\mathcal{U}$ leads to a problem $CO(\mathcal{U})$ that is, more often than not, harder than the deterministic problem CO. This is the case, for instance, when $\mathcal{U}$ is an

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arbitrary ellipsoid [7] or a set of two arbitrary scenarios [6]. Robust combinatorial optimization witnessed a breakthrough with the introduction of budgeted uncertainty in [8], which keeps the tractability of the deterministic counterpart for a large class of combinatorial optimization problems. Specifically, Bertsimas and Sim considered uncertain cost functions characterized by the vector $c \in \mathbb{R}^n$ of nominal costs and the vector $d \in \mathbb{R}^n_+$ of deviations. Then, given a budget of uncertainty $\Gamma > 0$, they addressed

$$CO_{\boldsymbol{d}}(\mathcal{U}_\Gamma) \qquad \min_{x\in\mathcal{X}} \max_{\xi\in\mathcal{U}_\Gamma} \sum_{i\in N}(c_i+\xi_i d_i)x_i = \min_{x\in\mathcal{X}}\left(\sum_{i\in N} c_i x_i + \max_{\xi\in\mathcal{U}_\Gamma}\sum_{i\in N}\xi_i d_i x_i\right),$$

for the budgeted uncertainty set $\mathcal{U}_\Gamma := \left\{\xi : \sum_{i\in N}\xi_i \le \Gamma, 0 \le \xi_i \le 1, i \in N\right\}$. Bertsimas and Sim [8] proved two fundamental results:

Theorem 1 ([8]). *Problem $CO_{\boldsymbol{d}}(\mathcal{U}_\Gamma)$ can be solved by solving $n+1$ problems CO with modified costs.*

Theorem 2 ([8]). *If CO admits a polynomial-time $(1+\epsilon)$-approximation algorithm running in $O(f(n,\epsilon))$, then $CO_{\boldsymbol{d}}(\mathcal{U}_\Gamma)$ admits a polynomial-time $(1+\epsilon)$-approximation algorithm running in $O(nf(n,\epsilon))$.*

These positive complexity results have been extended up to some extent to optimization problem with integer variables (i.e. $\mathcal{X} \subseteq \mathbb{Z}^n$) and constraints uncertainty in [9,10].

Another popular uncertainty model involves ellipsoids, and more particularly, axis-parallel ellipsoids, which we represent here through the robust counterpart

$$CO_{\boldsymbol{d}}(\mathcal{U}_{ball}) \qquad \min_{x\in\mathcal{X}}\left(\sum_{i\in N} c_i x_i + \max_{\xi\in\mathcal{U}_{ball}}\sum_{i\in N}\xi_i d_i x_i\right),$$

where c now represents the center of the ellipsoid, d gives the length of its axes, and $\mathcal{U}_{ball}$ is a ball of radius Ω, $\mathcal{U}_{ball} := \{\xi : \|\xi\|_2 \le \Omega\}$. Nikolova [11] proposes a counterpart of Theorem 2 for $CO_{\boldsymbol{d}}(\mathcal{U}_{ball})$ with a running time slightly worse than $O(\frac{1}{\epsilon^2}f(n,\epsilon))$. Her approach considers the problem as a two-objective optimization problem and approximates its pareto front. Other authors have addressed problem $CO_{\boldsymbol{d}}(\mathcal{U}_{ball})$, including Mokarami and Hashemi [12] who showed how the problem can be solved exactly by solving a pseudo-polynomial number of problems CO and [13,14] who provide polynomial special cases. A drawback of $CO_{\boldsymbol{d}}(\mathcal{U}_{ball})$ from the practical viewpoint is that $\mathcal{U}_{ball}$ contains vectors with high individual values. For that reason, a popular variation considers instead the uncertainty set defined as the intersection of a ball and a box, formally defined as $\mathcal{U}^{box}_{ball} := \left\{\xi : \|\xi\|_2 \le \Omega, -\underline{\xi} \le \xi \le \overline{\xi}\right\}$, for some $\underline{\xi}, \overline{\xi} \in \mathbb{R}^n_+$. While $\mathcal{U}^{box}_{ball}$ has been used in numerous papers dealing with robust optimization problems (e.g. [15,16]), we are not aware of previous complexity results for $CO_{\boldsymbol{d}}(\mathcal{U}^{box}_{ball})$.

The main focus of this paper is to study robust combinatorial optimization problem for uncertainty polytopes defined by bounds restrictions and $s = |S|$ knapsack constraints, specifically

$$\mathcal{U}_{knap} := \left\{\xi \in \mathbb{R}^n : \sum_{i\in N} a_{ji}\xi_i \le b_j, j \in S, 0 \le \xi \le \overline{\xi}\right\}, \tag{2}$$

where $a \in \mathbb{R}^{s\times n}_+$, $b \in \mathbb{R}^s_+$, and $\overline{\xi} \in \mathbb{R}^n_+$. Our definition $\mathcal{U}_{knap}$ is slightly more general than the multidimensional knapsack-constrained uncertainty set introduced in [17,18] since we consider non-negative values for the constraint coefficients while [17,18] assume all of them equal to 1. The author of [17,18] motivates the introduction of these complex polytopes in the context of multistage decision problems, where one wishes to correlate the value of uncertain parameters of a given period to those related to the precedent periods.

We relate next $\mathcal{U}_{knap}$ with the uncertainty polytopes that have been used in the literature for specific applications. Whenever $s = 1$, the resulting family of polytopes generalizes the uncertainty set