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The use of action functionals within the quantum-like paradigm

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HIGHLIGHTS

- Arbitrage can be characterized by a measure of curvature.
- The Fourier transform can be used to show how a pdf on the amplitude function of wave numbers (they now depend on ε) yields a pdf.
- We can source arbitrage based risk neutral probabilities from this pdf.
- Via the route of the Fourier transform, we can also introduce some quantum-like ideas in economics.
- Superposed values of a good could be seen to be equal to an unobserved, agent based price.

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ABSTRACT

Arbitrage is a key concept in the theory of asset pricing and it plays a crucial role in financial decision making. The concept of the curvature of so-called 'fibre bundles' can be used to define arbitrage. The concept of 'action' can play an important role in the definition of arbitrage. In this paper, we connect the probabilities emerging from a (non) zero linear action with so-called risk neutral probabilities. The paper also shows how arbitrage/non arbitrage can be well defined within a quantum-like paradigm. We also discuss briefly the behavioural dimension of arbitrage.

1. Introduction

The concept of 'arbitrage' is of paramount importance in the theory of asset pricing. It also plays an important role in behavioural economics, since the real (or even imagined) presence of arbitrage possibilities has a powerful influence on the psychology of agents of the financial market.[1] In essence an arbitrage opportunity implies that a positive financial return can be realized, which is in excess of the risk free rate of interest,[2] on taking a trading position in an asset which entails no financial risk. Disregarding the cost which may need to be incurred for finding such arbitrage opportunities, one could in effect claim that an arbitrage opportunity is akin to obtaining what is often quoted in common parlance as a 'free lunch'. It is intuitive that pricing financial assets under the assumption there does not exist arbitrage, rationalizes the existence of so-called 'benchmark' asset prices. As Øksendal (2004) indicates "It is not possible to make a sensible mathematical theory for a market with arbitrage" (p. 26). As an example Black–Scholes option pricing theory, which commands a market of trillions of dollars, rests on the assumption of no-arbitrage.[3]

The conditions for no-arbitrage to occur for a discrete parameter process can be found in Harrison and Kreps (1979). It is important to make the distinction on the required conditions for non-arbitrage, between discrete and continuous parameter processes. As mentioned in Karatzas and Schreve (1998): "if one cannot win for certain by betting on a given process" (p. 33), i.e. one cannot make a riskless profit, then under a discrete parameter

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[1] See Shleifer and Vishny (1997) and references on various psychological distortions in the behaviour of arbitrageurs and their clients and global financial consequences of such psychological particularities.

[2] The risk free rate of return is the return which is deemed to exist on financial assets which have no financial risk.

[3] The theory assumes a geometric Brownian motion of asset prices and such motion process invites the use of a calculus, also known under the name of 'Ito calculus'. A while ago, the mathematical finance literature considered a departure from the above Brownian motion, by considering a so-called fractional Brownian motion. Such motion is characterized by a degree of memory (this refers to the so-called Hurst exponent). Ito integration in that context is then replaced by a so-called Wick integral (Björk & Hult, 2005). Björk and Hult (2005) also report on how one can (not) make arguments that such fractional Brownian motion is arbitrage free.

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process most often such process will be a martingale.[4] However, this is not necessarily the case when a continuous parameter process is considered. Asset pricing is hinging very much so, on the concept of an equivalent martingale which in essence refers to the use of a probability which converts a semi-martingale into a martingale. Karatzas and Schreve (1998) do mention that this type of equivalent measure "bear(s) a striking similarity to de Finetti's (1937, 1974) theory of coherent subjective probabilities and inferences..." (p. 34). We remark that in Haven and Khrennikov (in press), we discuss subjective interpretations of probability within a quantum-like environment. From the outset, we note that since this paper will consider the notion of 'quantum-likeness', we can as well make an attempt to describe what is meant with this novel term. In Khrennikov and Haven (2016), we mention in the preface that in fact it is the preponderance of the field of quantum information, which puts to the fore the interpretation that the wave function, a central object in quantum mechanics, is informational in nature. As we explain, the formalism of quantum mechanics can be used to describe information processing of any system, whether social or physical, with the caveat that there must be some sort of quantum feature to the system under study. We need to be careful with what we mean with "quantum feature". The utilization of wave functions, or the use of analogies with an uncertainty principle can (but does not have to) invoke quantum features. Wave functions can be found in classical mechanics. The idea of an uncertainty principle (for instance) exists in electrical engineering (via the time–frequency uncertainty principle). Where the term "quantum feature" has more traction, so to speak, is in the area where quantum probability (in decision making) is employed. In that same area do we find the difficult concept of 'context' (Khrennikov, 2010). Quantum features occur also in the use of Fisher information in economics and finance. Fisher information is narrowly linked to a particular potential function which emerges from quantum mechanics (see Hawkins & Frieden, 2012).

We also note that quantum-like models describing decision making, when in particular such decision making occurs at the level of the financial market, will not match the canonical understanding of rationality (based on Savage's Bayesian approach). Quantum-like decision makers are irrational (from the viewpoint of the classical theory of rationality). At the same time they are completely rational from the quantum-like viewpoint where rational behaviour corresponds to the non-Bayesian updating of probabilities, which is mathematically represented by the quantum probability calculus (Khrennikov, 2015). As was reflected in a number of publications on the theoretical analysis of arbitrage, the success and failure of arbitrageurs is closely coupled to a degree of irrationality (in the classical sense) of traders of the financial market and especially their clients (in the case arbitrageurs do not operate with their own money).

Within the context of gauge field dynamics, Ilinski (2001) provides for a very elegant approach towards the formulation of arbitrage. He uses the theory of fibre bundles and the curvature of such fibre bundles to characterize the degree of arbitrage. The theory of fibre bundles can be found in Steenrod (1951) and Bishop and Crittenden (1964). We will be very brief on this theory. However, we will be particularly interested in seeing how the concept of 'action', Ilinski (2001) introduces, can operate in a non-arbitrage context. We will therefore be keen in linking the role of action to the existence of risk neutral probabilities and state prices. Finally, a major objective of this paper will consist in characterizing arbitrage within a Fourier integral setting.

The outline of the paper is as follows. In the next section, following Ilinski (2001) we briefly introduce the notion of a fibre bundle and we attempt to show how curvature relates to the concept of arbitrage. In the section following we consider the role of linear action in the context of the (non) existence of arbitrage. We then argue how such action plays a role in risk neutral probabilities and state price formation. Section 5 of the paper introduces the idea that the risk neutral probabilities (whether they exist under the assumption arbitrage occurs or not) can also be derived from a density function generated via a Fourier integral. We will be careful in defining the precise (economics-based) ingredients of this Fourier integral. Sections 6 and 7 provide for two applications. Those two applications also go into quite some detail in showing how interaction, arbitrage and the Fourier integral may be linked together. Section 8 summarizes the two applications and we conclude the paper in Section 9.

2. Fibre bundles and curvature

2.1. A short introduction

Fibre bundles are in fact quite easy objects to define in plain language. Since this paper does not consider fibre bundles as the central object of study, we will introduce the concepts in an as simple way as possible. Ilinski (2001) says that "each fibre bundle consists of identical subspaces that are all collected together... to give the whole space" (p. 19). Ilinski (2001, p. 21) shows a simple example of a tube which can be seen as a fibre bundle with a circular base and line fibres. A more involved example may consist of a fibre bundle which is a "spherical base and the fibres which are two dimensional tangent planes stuck to each point of the sphere." (Ilinski, 2001, p. 25). We note that the fibre bundle is known under other names such as 'fibration'; 'twisted product' or also 'Steenrod bundle' (see Monastyrsky, 1993, p. 48). Thus, we can intuitively grasp (in a very informal way) that a fibre bundle can be a manifold with a base which is a manifold. Fibres, F, which make up the fibre bundle are also manifolds. Now, one can imagine a rule for the 'movement' of an element of the fibre, say x, from one point of the base to another point of the base, say y. An operator, $U(\gamma)$ can be defined which describes a 'movement' along some curve γ; $U(\gamma) : F_x \to F_y$. See below for more details (especially Definition 5).

2.2. Fibre bundle operators and curvature

Ilinski (2001) was the first to introduce and fine tune the use of fibre bundles in an economics context. The objective of this subsection is to briefly describe a simple example where the curvature idea can be easily explained. Examples 1 and 2; Claims 3 and 4 and Definitions 5 and 6; and Example 7 follow Ilinski (2001, p. 88–90).

Example 1. Assume we have two assets (i and $i+1$): cash and a share of a stock of some company. Then define $U((i, n), (i, n+1)) = e^{r_1 \Delta}$; where r_1 is the return on a share of a company and Δ is the equally spaced time difference (between times $n+1$ and n). Hence, if one has 1 unit of currency to invest in a company share over a period of time Δ then one obtains (continuously compounded): $e^{r_1 \Delta}$. Now define $U((i+1, n), (i+1, n+1)) = e^{r_0 \Delta}$, where r_0 is the return on cash and Δ is the equally spaced time difference. The inverse movement can be defined too: $U((i+1, n+1), (i+1, n)) = e^{-r_0 \Delta}$ (this is the present value of 1 unit of currency). Similarly for the inverse movement, relative to $U((i, n), (i, n+1))$, which is $U((i, n+1), (i, n)) = e^{-r_1 \Delta}$. Now consider $U((i, n), (i+1, n)) = S_i$, where S_i is the price in unit of currency for the share of a company (buy a share). One share is thus exchanged on S_i units of cash at some point in time t_i. Define

4 Roughly, a martingale can be defined as a conditional expectation of a variable S_{t+1} (or a process of variables), where $t+1$ indicates the future given the information available up to t, F_t, such that $E(S_{t+1}|F_t) = S_t$. A semi-martingale requires that either $E(S_{t+1}|F_t) > S_t$ or $E(S_{t+1}|F_t) < S_t$.