



# A multiscale extension of the Margrabe formula under stochastic volatility



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## ABSTRACT

The pricing of financial derivatives based on stochastic volatility models has been a popular subject in computational finance. Although exact or approximate closed form formulas of the prices of many options under stochastic volatility have been obtained so that the option prices can be easily computed, such formulas for exchange options leave much to be desired. In this paper, we consider two different risky assets with two different scales of mean-reversion rate of volatility and use asymptotic analysis to extend the classical Margrabe formula, which corresponds to a geometric Brownian motion model, and obtain a pricing formula under a stochastic volatility. The resultant formula can be computed easily, simply by taking derivatives of the Margrabe price itself. Based on the formula, we show how the stochastic volatility corrects the Margrabe price behavior depending on the moneyness and the correlation coefficient between the two asset prices.

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## 1. Introduction

There is a well known formula called the Margrabe formula [1] which is a closed form pricing formula for European options to exchange one risky asset for another at maturity. The prices of two underlying assets in this formula are assumed to follow geometric Brownian motions (GBMs). The volatilities of these assets may not be necessarily constant but the volatility of the ratio of the two assets has to be constant. Although it is a nice starting point to price an exchange option based on a GBM for the underlying asset, it is well-known to give rise to a controversy. For example, the flat implied volatility of the model would not capture the volatility smile or skew which is well observed in the market. Also, the underestimation of extreme movement in risky asset prices yielding tail risk is another drawback of the GBM assumption.

In general, there have been lots of efforts to overcome the limit of the GBM process in quantitative finance. The constant elasticity of variance model developed by Cox [2], a stochastic volatility model by Heston [3] or Fouque et al. [4], and a Levy model by Carr et al. [5] are quite popular as they are taking center stage in the current mathematical finance community. And there are many interesting modified versions including the ones given by Bonanno et al. [6] and [7]. In terms of exchange option, however, studies

of the Margrabe formula based on these advanced models leave much to be desired. There are a few number of works reported so far regarding the exchange option. Bernard et al. [8] considered the Margrabe formula with stochastic interest rates, Antonelli et al. [9] studied exchange option pricing under stochastic volatility via a correlation expansion, and Alos et al. [10] extended the Margrabe formula under general stochastic volatility.

This paper has chosen a stochastic volatility model with fast mean-reversion which was developed by Fouque et al. [4] in order to extend the Margrabe formula and investigate stochastic volatility effect on the formula. While the study of Antonelli et al. [9] has assumed that the correlation parameters are small enough to make a possible Taylor series expansion and Alos et al. [10] has assumed that the volatility processes of two assets are driven by the same Brownian motion (perfect correlation), our study is based on the assumption that they are independent but they are both fast mean-reverting with two different speeds of the mean-reversion. Since the fast mean-reversion is a stylized fact for stochastic volatility, our specification is not believed to be restrictive. If the two Brownian motions driving the volatilities are assumed to be the same Brownian motion, then it would add an extra term to our result without harming the methodology. However, the focal point of this paper is the study of stochastic volatility effects driven by the multi-scale property of the mean-reversion. The biggest advantage of our formulation is that the corresponding formula is explicitly given by the Greeks of the classical Margrabe formula so that its computation can become much easier.

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This paper is structured as follows. In Section 2, we formulate a stochastic volatility model for an exchange option and obtain a singularly perturbed partial differential equation problem for the option price. In Section 3, we use asymptotic expansion method to obtain an explicit formula for the solution of it. Section 4 discloses the stochastic volatility effect on the Margrabe formula. Section 5 concludes.

## 2. Model formulation

We consider two risky assets whose prices are denoted by  $S_t^1$  and  $S_t^2$ , respectively. Based on the fast mean-reverting stochastic volatility model of Fouque et al. [4] for a single asset, let  $S_t^1$  and  $S_t^2$  satisfy the stochastic differential equations

$$dS_t^1 = rS_t^1 dt + f(Y_t^1)S_t^1 dW_t^1, \\ dY_t^1 = \left( \frac{1}{\epsilon} (m_1 - Y_t^1) - \frac{v_1 \sqrt{2}}{\sqrt{\epsilon}} \Lambda^1(Y_t^1) \right) dt + \frac{v_1 \sqrt{2}}{\sqrt{\epsilon}} dZ_t^1, \quad (1)$$

$$dS_t^2 = rS_t^2 dt + g(Y_t^2)S_t^2 dW_t^2, \\ dY_t^2 = \left( \frac{1}{\delta} (m_2 - Y_t^2) - \frac{v_2 \sqrt{2}}{\sqrt{\delta}} \Lambda^2(Y_t^2) \right) dt + \frac{v_2 \sqrt{2}}{\sqrt{\delta}} dZ_t^2, \quad (2)$$

respectively, where  $r$  is a risk-free interest rate,  $\Lambda^1$  and  $\Lambda^2$  are the market prices of volatility risk,  $\epsilon$  and  $\delta$  are sufficiently small positive parameters corresponding to the short mean-reversion time scale of the volatility factor  $Y_t^1$  and  $Y_t^2$ , the functions  $f$  and  $g$  are volatility functions driven by the processes  $Y_t^1$  and  $Y_t^2$ , respectively, which are positive and smooth functions such that  $f^2$  and  $g^2$  are integrable with respect to the invariant distribution of  $Y_t^1$  and  $Y_t^2$ , respectively. And  $W_t^1$ ,  $W_t^2$ ,  $Z_t^1$  and  $Z_t^2$  are Brownian motions whose correlation structure is given by  $dW_t^1 dZ_t^1 = \rho_{11} dt$ ,  $dW_t^1 dW_t^2 = \rho_{12} dt$  and  $dW_t^2 dZ_t^2 = \rho_{22} dt$  and zero otherwise. We note that the process  $Y_t^i$  is an ergodic process having an invariant distribution, denoted by  $\Phi_i$ , which is normal with  $N(m_i, v_i)$ ,  $i = 1, 2$ .

The price at time  $t$  of an option to exchange asset 2 for asset 1 is defined by

$$P(t, s_1, s_2, y_1, y_2; T) = E[e^{-r(T-t)} \max(S_T^1 - S_T^2, 0) | S_t^1 = s_1, S_t^2 = s_2, Y_t^1 = y_1, Y_t^2 = y_2]$$

with a final condition given by  $P(T, s_1, s_2, y_1, y_2; T) = \max(s_1 - s_2, 0)$ . Since the 4-dimensional joint process  $(S_t^1, S_t^2, Y_t^1, Y_t^2)$  is a Markov process, the Feynman–Kac theorem (cf. Oksendal [11]) leads to a partial differential equation problem given by

$$\left( \frac{1}{\epsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}} \mathcal{L}_1 + \mathcal{L}_{ex} + \frac{1}{\delta} \mathcal{M}_0 + \frac{1}{\sqrt{\delta}} \mathcal{M}_1 \right) P(t, s_1, s_2, y_1, y_2; T) = 0, \\ t < T, \quad (3)$$

$$P(T, s_1, s_2, y_1, y_2; T) = \max(s_1 - s_2, 0), \quad (4)$$

where the operators  $\mathcal{L}_0$ ,  $\mathcal{L}_1$ ,  $\mathcal{L}_{ex}$ ,  $\mathcal{M}_0$ , and  $\mathcal{M}_1$  are given by

$$\mathcal{L}_0 = (m_1 - y_1) \frac{\partial}{\partial y_1} + v_1^2 \frac{\partial^2}{\partial y_1^2}, \\ \mathcal{L}_1 = \rho_{11} f(y_1) v_1 s_1 \sqrt{2} \frac{\partial^2}{\partial s_1 \partial y_1} - \sqrt{2} v_1 \Lambda^1(y_1) \frac{\partial}{\partial y_1}, \\ \mathcal{L}_{ex} = \frac{\partial}{\partial t} + \frac{1}{2} f(y_1)^2 s_1^2 \frac{\partial^2}{\partial s_1^2} + \frac{1}{2} g(y_2)^2 s_2^2 \frac{\partial^2}{\partial s_2^2} \\ + \rho_{12} f(y_1) g(y_2) s_1 s_2 \frac{\partial^2}{\partial s_1 \partial s_2} + r \left( s_1 \frac{\partial}{\partial s_1} + s_2 \frac{\partial}{\partial s_2} - \cdot \right), \\ \mathcal{M}_0 = (m_2 - y_2) \frac{\partial}{\partial y_2} + v_2^2 \frac{\partial^2}{\partial y_2^2},$$

$$\mathcal{M}_1 = \rho_{22} g(y_2) v_2 s_2 \sqrt{2} \frac{\partial^2}{\partial s_2 \partial y_2} - \sqrt{2} v_2 \Lambda^2(y_2) \frac{\partial}{\partial y_2}, \quad (5)$$

respectively. This is a singularly perturbed partial differential equation problem. It can not be solved for general functions  $f$  and  $g$ . So, we use asymptotic expansion method of Fouque et al. [4] to obtain an approximate solution of it.

## 3. Multiscale asymptotic expansion

### 3.1. Leading order term

Under the assumption  $\epsilon \ll \delta \ll \sqrt{\epsilon}$ , we expand the option price  $P$  formally as

$$P = P_{0,0} + \sqrt{\delta} P_{0,1} + \sqrt{\epsilon} P_{1,0} + \delta P_{0,2} + \sqrt{\epsilon} \delta P_{1,1} + \epsilon P_{2,0} + \delta \sqrt{\delta} P_{0,3} \\ + \delta \sqrt{\epsilon} P_{1,2} + \epsilon \sqrt{\delta} P_{2,1} + \epsilon \sqrt{\epsilon} P_{3,0} + \dots,$$

where we assume that each term  $P_{i,j}$  does not grow exponentially in  $y_1^2$  and  $y_2^2$ .

Substituting this expansion into Eq. (3), we obtain a hierarchy of partial differential equations in the descending order of  $\frac{1}{\epsilon}$ ,  $\frac{1}{\delta}$ ,  $\frac{\sqrt{\delta}}{\epsilon}$ ,  $\frac{1}{\sqrt{\epsilon}}$ ,  $\frac{1}{\sqrt{\delta}}$ ,  $\frac{\sqrt{\epsilon}}{\delta}$ ,  $\frac{\delta}{\epsilon}$ , and etc as follows.

$$\frac{1}{\epsilon} (\mathcal{L}_0 P_{0,0}) + \frac{1}{\delta} (\mathcal{M}_0 P_{0,0}) + \frac{\sqrt{\delta}}{\epsilon} (\mathcal{L}_0 P_{0,1}) \\ + \frac{1}{\sqrt{\epsilon}} (\mathcal{L}_1 P_{0,0} + \mathcal{L}_0 P_{1,0}) + \frac{1}{\sqrt{\delta}} (\mathcal{M}_1 P_{0,0} + \mathcal{M}_0 P_{0,1}) \\ + \frac{\sqrt{\epsilon}}{\delta} (\mathcal{M}_0 P_{1,0}) + \frac{\delta}{\epsilon} (\mathcal{L}_0 P_{0,2}) + \frac{\sqrt{\delta}}{\sqrt{\epsilon}} (\mathcal{L}_1 P_{0,1} + \mathcal{L}_0 P_{1,1}) \\ + (\mathcal{L}_0 P_{2,0} + \mathcal{M}_0 P_{0,2} + \mathcal{L}_1 P_{1,0} + \mathcal{M}_1 P_{0,1} + \mathcal{L}_{ex} P_{0,0}) \\ + \frac{\sqrt{\epsilon}}{\sqrt{\delta}} (\mathcal{M}_1 P_{1,0} + \mathcal{M}_0 P_{1,1}) + \frac{\epsilon}{\delta} (\mathcal{M}_0 P_{2,0}) \\ + \frac{\delta}{\epsilon} \sqrt{\delta} (\mathcal{L}_0 P_{0,3}) + \frac{\delta}{\sqrt{\epsilon}} (\mathcal{L}_1 P_{0,2} + \mathcal{L}_0 P_{1,2}) \\ + \sqrt{\delta} (\mathcal{M}_0 P_{0,3} + \mathcal{M}_1 P_{0,2} + \mathcal{L}_{ex} P_{0,1} + \mathcal{L}_1 P_{1,1} + \mathcal{L}_0 P_{2,1}) \\ + \sqrt{\epsilon} (\mathcal{L}_0 P_{3,0} + \mathcal{L}_1 P_{2,0} + \mathcal{L}_{ex} P_{1,0} + \mathcal{M}_1 P_{1,1} + \mathcal{M}_0 P_{1,2}) \\ + \frac{\epsilon}{\sqrt{\delta}} (\mathcal{M}_1 P_{2,0} + \mathcal{M}_0 P_{2,1}) + \frac{\epsilon \sqrt{\epsilon}}{\delta} (\mathcal{M}_0 P_{3,0}) + \dots = 0. \quad (6)$$

First, from the  $O(\frac{1}{\epsilon})$  term of (6) we have  $\mathcal{L}_0 P_{0,0} = 0$  which yields the  $y_1$ -independence of  $P_{0,0}$  since  $P_{0,0}$  does not grow exponentially in  $y_1^2$ . Also, we obtain the  $y_2$ -independence of  $P_{0,0}$  from  $\mathcal{M}_0 P_{0,0} = 0$  which is the  $O(\frac{1}{\delta})$  term of (6). From the  $O(\frac{\sqrt{\delta}}{\epsilon})$  term,  $P_{0,1}$  is independent of  $y_1$ . From the  $O(\frac{1}{\sqrt{\epsilon}})$  term,  $\mathcal{L}_1 P_{0,0} + \mathcal{L}_0 P_{1,0} = 0$  which leads to  $\mathcal{L}_0 P_{1,0} = 0$  because of  $y_1$ -independence of  $P_{0,0}$ . So,  $P_{1,0}$  is independent of  $y_1$ . In this way, one can obtain the  $y_2$ -independence of  $P_{0,1}$  and  $P_{1,0}$  from the  $O(\frac{1}{\sqrt{\delta}})$  and  $O(\frac{\sqrt{\epsilon}}{\delta})$  terms, respectively. Hence,  $P_{0,0}$ ,  $P_{0,1}$  and  $P_{1,0}$  are all independent of  $y_1$  as well as  $y_2$ .

On the other hand, the  $y_1$ -independence of  $P_{0,2}$  and the  $y_2$ -independence of  $P_{2,0}$  follows, respectively, from the  $O(\frac{\delta}{\epsilon})$  and  $O(\frac{\epsilon}{\delta})$  terms of (6).

Based the independence result above, we now try to obtain an equation for the leading order term  $P_{0,0}$ . First, from the  $O(1)$  term, the  $y_1, y_2$ -independency of  $P_{1,0}$  and  $P_{0,1}$  gives rise to

$$\mathcal{L}_0 P_{2,0} + \mathcal{M}_0 P_{0,2} + \mathcal{L}_{ex} P_{0,0} = 0.$$

Since this can be regarded as a Poisson equation for  $P_{2,0}$  or  $P_{0,2}$  with respect to  $\mathcal{L}_0$  or  $\mathcal{M}_0$ , respectively, by applying the centering condition (for the existence of solutions) from the Fredholm alternative (cf. Ramm [13]) to the equation, we obtain

$$\mathcal{M}_0 P_{0,2} + \langle \mathcal{L}_{ex} \rangle^{y_1} P_{0,0} = 0,$$

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