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Optimal time-consistent investment and reinsurance policies for mean-variance insurers☆

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ABSTRACT

This paper investigates the optimal time-consistent policies of an investment-reinsurance problem and an investment-only problem under the mean-variance criterion for an insurer whose surplus process is approximated by a Brownian motion with drift. The financial market considered by the insurer consists of one risk-free asset and multiple risky assets whose price processes follow geometric Brownian motions. A general verification theorem is developed, and explicit closed-form expressions of the optimal polices and the optimal value functions are derived for the two problems. Economic implications and numerical sensitivity analysis are presented for our results. Our main findings are: (i) the optimal time-consistent policies of both problems are independent of their corresponding wealth processes; (ii) the two problems have the same optimal investment policies; (iii) the parameters of the risky assets (the insurance market) have no impact on the optimal reinsurance (investment) policy; (iv) the premium return rate of the insurer does not affect the optimal policies but affects the optimal value functions; (v) reinsurance can increase the mean-variance utility.

1. Introduction

Insurers can control their risks by means of some business activities, such as investing in a financial market, purchasing reinsurance, and acquiring new business (acting as a reinsurer for other insurers). As a result, there have arisen many optimization problems with various objectives in insurance risk management. This topic has been extensively investigated in the literature. For example, Browne (1995) obtains the optimal investment strategies for an insurer who maximizes the expected utility of the terminal wealth or minimizes the ruin probability, where the surplus process of the insurer is modeled by a drifted Brownian motion. Yang and Zhang (2005) study the optimal investment policies for an insurer who maximizes the expected exponential utility of the terminal wealth or maximizes the survival probability, where the surplus process is driven by a jump–diffusion process. Further, Xu et al. (2008), Cao and Wan (2009) and Gu et al. (2010) study the optimal investment-reinsurance policies for an insurer who maximizes the expected utility of the terminal wealth in different situations.

Recently, many scholars consider the optimal investment and/or reinsurance policies for insurers under the mean-variance criterion, which is pioneered by Markowitz (1952) and has long been recognized as the milestone of modern portfolio theory. For example, Bäuerle (2005) considers the optimal proportional reinsurance/new business problem under the mean-variance criterion where the surplus process is modeled by the classical Cramér–Lundberg (CL) model, and derives the optimal policy in closed-form. Delong and Gerrard (2007) consider two optimal investment problems for an insurer: one is the classical mean-variance portfolio selection and the other is the mean-variance terminal objective involving a running cost penalizing deviation of the insurer's wealth from a specified profit-solvency target. They assume that the claim process is a compound Cox process with the intensity described by a drifted Brownian motion and the insurer invests in a financial market consisting of a risk-free asset and a risky asset whose price is driven by a Lévy process. Bai and Zhang (2008) study the optimal investment-reinsurance policies for an insurer under the mean-variance criterion by the linear quadratic

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(LQ) method and the dual method, where they assume that the surplus of the insurer is described by a CL model and a diffusion approximation (DA) model respectively. Zeng et al. (2010) assume that the surplus of an insurer is modeled by a jump–diffusion process, and derive the optimal investment policies explicitly under the benchmark and mean-variance criteria by the stochastic maximum principle.

It is apparent to all that the mean-variance criterion lacks the iterated-expectation property, which results in that continuous-time/multi-period mean-variance problems are time-inconsistent in the sense that the Bellman Optimality Principle does not hold and hence the traditional dynamic programming approach cannot be directly applied. The optimal policies to dynamic mean-variance problems considered in all the literature mentioned above are derived under the implicit assumption that the decision makers pre-commit themselves to follow in the future the policies chosen at the initial time, namely, the decision makers initially choose policies to maximize their objective functions at time 0 and thereafter do not deviate from these policies. Such policies are so-called pre-commitment policies, which are time-inconsistent in that they are optimal only when sitting at the initial time.

However, time consistency of policies is a basic requirement for rational decision making in many situations. A decision maker sitting at time t would consider that, starting from $t + \Delta t$, she will follow the policy that is optimal sitting at time $t + \Delta t$. Namely, the optimal policy derived at time t should agree with the optimal policy derived at time $t + \Delta t$. Strotz (1956) first analytically formalizes time inconsistency and works on time-consistent policies for time-inconsistent problems. He proposes that time-inconsistent problems can be solved either by pre-commitment policies or by time-consistent policies. In very recent times, time-inconsistent stochastic control problems have attracted much attention. Bjök and Murgoci (2009) develop a general theory for Markovian time-inconsistent stochastic control problems with fairly general objectives. They derive an extension of the standard Hamilton–Jacobi–Bellman (HJB) equation in the form of a system of non-linear PDFs. Wang and Forsyth (submitted for publication) study the time-consistent policy and the pre-commitment policy of a continuous-time mean-variance asset allocation problem and develop a numerical scheme which can determine the optimal policy whatever type of constraint is applied to the investment behavior. Bjök et al. (2010) consider a mean-variance portfolio optimization problem with state-dependent risk aversion in a continuous-time setting. Basak and Chabakauri (2010) study a dynamic mean-variance asset-allocation problem within a Wiener driven framework and derive the explicit time-consistent policy by solving the extended HJB equation.

As far as we know, there is no literature on the optimal investment and reinsurance problems for mean-variance insurers who are concerned about the time-consistent policies. In this paper we try to pioneer this study. Specifically, we consider the optimal time-consistent policies of an investment-reinsurance problem and an investment-only problem for a mean-variance insurer. In the first problem, the insurer is allowed to invest in a financial market and purchase proportional reinsurance/acquire new business. In the second problem, the insurer is only allowed to invest in a financial market but not allowed to purchase proportional reinsurance/acuqire new business. In both problems, the insurer is of mean-variance preference, the surplus process of the insurer is modeled by a DA model, and the financial market consists of one risk-free asset and multiple risky assets whose price processes are driven by geometric Brownian motions. We develop a general verification theorem and derive closed-form expressions for the optimal time-consistent policies and the optimal value functions of the two problems. We also present economic implications of our results and provide sensitivity analysis by a numerical example.

The rest of this paper is organized as follows. Section 2 describes the model and some assumptions. Section 3 formulates the optimization problems and gives a general verification theorem. The investment-reinsurance problem and the investment-only problem under the mean-variance criterion without pre-commitment are solved in Section 4. Section 5 provides a numerical sensitivity analysis and Section 6 concludes the paper.

2. Model and assumptions

We start with a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, \mathcal{P})$, where T is a finite and positive constant, representing the time horizon, $\mathcal{F}_t$ stands for the information available at time t, and any decision made at time t is based upon such information. All stochastic processes introduced below are supposed to be well-defined and adapted processes in this space.

2.1. Surplus process

We consider an insurer whose surplus process is modeled by a DA model. To understand the DA model better, it is advantageous to start from the classical CL model. In the CL model the claims arrive according to a homogeneous Poisson process $\{N_t\}$ with intensity λ; the individual claim sizes are $Z_i,\ i = 1, 2, \ldots,$ which are assumed to be independent of $\{N_t\}$ and be independent and identically distributed (i.i.d.) positive random variables with finite first and second-order moments given by μ_∞ and σ_∞^2, respectively. Then the surplus process of the insurer without reinsurance and investment follows

$$\mathrm{d}R(t) = c\mathrm{d}t - \mathrm{d}\sum_{i=1}^{N_t} Z_i, \tag{1}$$

where c is the premium rate which is assumed to be calculated according to the expected value principle, i.e., $c = (1+\eta)\lambda\mu_\infty$, and here $\eta > 0$ is the relative safety loading of the insurer. By Grandll (1991), the CL model can be approximated by the following diffusion model

$$\mathrm{d}R(t) = \mu\mathrm{d}t + \sigma_0\mathrm{d}W_0(t), \tag{2}$$

where $\mu = \eta\lambda\mu_\infty$ can be regarded as the premium return rate of the insurer, $\sigma_0^2 = \lambda\sigma_\infty^2$ measures the volatility of the insurer's surplus, $\{W_0(t)\}$ is a standard Brownian motion. It is worth pointing out that the DA model (2) works well for large insurance portfolios, where an individual claim is relatively small compared to the size of surplus. The DA model has been used in much existing literature, for example, Browne (1995), Promislow and Young (2005), Gerber and Shiu (2006), Bai and Guo (2008), Cao and Wan (2009), Chen et al. (2010), Gu et al. (2010), and so on.

In addition, the insurer is allowed to purchase proportional reinsurance or acquire new business (for example, acting as a reinsurer of other insurers, see Bäuerle (2005)) at each moment in order to control insurance business risk. The proportional reinsurance/new business level is associated with the value of risk exposure $a(t) \in [0, +\infty)$ at any time $t \in [0, T]$. $a(t) \in [0, 1]$ corresponds to a proportional reinsurance cover and shows that the cedent should divert part of the premium to the reinsurer at the rate of $(1-a(t))\theta$, where θ can be regarded as the premium return rate of the reinsurer. Meanwhile, the insurer should pay $100a(t)\%$ while the rest $100(1 - a(t))\%$ is paid by the reinsurer for each claim occurring at time t. The proportional reinsurance is called cheap if $\theta = \mu$ while being not cheap if $\theta > \mu$. $a(t) \in (1, +\infty)$ corresponds to acquiring new business. For convenience, we call the process of risk exposure $\{a(t) : t \in [0, T]\}$ as a reinsurance policy. When a reinsurance policy $\{a(t) : t \in [0, T]\}$ is adopted, the corresponding DA dynamics for the surplus process becomes

$$\mathrm{d}R(t) = [\mu - (1 - a(t))\theta]\mathrm{d}t + \sigma_0 a(t)\mathrm{d}W_0(t). \tag{3}$$