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Robust simulation of rational discrete-time systems via sum of squares relaxations

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Abstract: This paper concerns the simulation of a class of nonlinear discrete-time systems under a set of initial conditions described by an ellipsoid. We derive a procedure allowing the propagation of such ellipsoids through time, which makes it possible to set a guaranteed hard bound on the evolution of the state of the system for all the possible initial conditions. At the end of the paper, we show an application of the method through three academics examples, two of which are taken from the theory of fractals.

Keywords: Discrete-time systems, linear matrix inequalities, sum of squares, robust simulation, fractals

1. INTRODUCTION

When dealing with nonlinear systems, it is often of capital importance to be able to predict the evolution of the state for uncertain initial condition. A practical approach to this problem is the use of systematic simulation, which consist ideally in checking the behaviour of the system with respect to all possible initial conditions; but this is strictly impossible if initial conditions are assumed to belong to a dense set, leading to approaches based on random tests or Monte Carlo methods (Binder [1986]). Other approaches have also been investigated, for example based on random exploration like in Donzé and Maler [2007] or sensitivity analysis as in Dang et al. [2008]. All of these methods alas suffer of the shortcoming of being a sort of "statistical" validation, in the sense that they do not offer a hard bound on the evolution of the system, if not for a dramatically increasing computational complexity. Another approach consists in evaluating the effect of the initial condition with respect to an output index, which gives an idea of such effects but does not establish precise bounds on each state variable (Tierno et al. [1997], Jönsson [2002]).

In this article we present a radically different approach to the problem, based on the so called "robust simulation" or simulation of sets (Kantner and Doyle [1996], Kishida and Braatz [2011], Topcu et al. [2008], Calafiore [2003], Ben-Talha et al. [(in press]), which offers instead mathematically guaranteed bounds for the evolution of dynamical systems. The specific method developed in this paper focuses on a class of discrete-time systems, and it is based on a relaxation of polynomial problems (Parrilo [2003]), which leads to efficiently solvable convex optimisation problems in the form of linear matrix inequalities (LMIs, Boyd et al. [1994]). This approach can be considered "safe", as the evolution of all the possible trajectories of the state are hard bounded, but on the other hand it is conservative, i.e. the bounds are not necessarily tight. Robust simulation is a problem that shares several similarities with many works on computations on invariant sets Korda et al. [2014], reachability Shia et al. [2014], Wang et al. [2013] and search for regions of attractions Henrion and Korda [2014]. The goal of this paper is to find the envelope containing the state *at each time instant*, and not to understand whether it is stable or whether it will eventually converge to a set.

The paper is organised as follows. Section 2 contains the preliminaries and the problem formulation. The main theoretical result can then be found in Section 3, in the form of a theorem with a corollary. This result is then applied to three examples in Section 4. Finally, conclusions are drawn in Section 5.

2. PRELIMINARIES

2.1 Notation

We denote by $\mathbb{N}$ the set of natural numbers, by $\mathbb{R}$ the set of real numbers and by $\mathbb{R}^{n\times m}$ the set of real $n \times m$ matrices. Let $\mathbb{C}$ be the set of complex numbers, and j the imaginary unit. $A^\top$ indicates the transpose of a matrix A, I_n is the identity matrix of size n, and $0_{n\times m}$ is a matrix of zeros of size $n \times m$. The notation $A \succeq 0$ ($A \preceq 0$) indicates that all the eigenvalues of the symmetric matrix A are positive (negative) or equal to zero, whereas $A \succ 0$ ($A \prec 0$) indicates that all such eigenvalues are strictly positive (negative). The symbol $\binom{n}{k}$ indicates the binomial coefficient, for which we have

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

We also define $\mathcal{E}(P, c)$ as the ellipsoid of dimension n with matrix $P \in \mathbb{R}^{n\times n}$, $P = P^\top \succ 0$ and centered in $c \in \mathbb{R}^n$,

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i.e. $\mathcal{E}(P,c) = \{x \in \mathbb{R}^n \,|\, (x-c)^\top P^{-1}(x-c) \leqslant 1\}$. At last, we employ the symbol $*$ to complete symmetric matrix expressions avoiding repetitions.

2.2 Problem formulation

We consider a discrete-time dynamical system of order n whose evolution is described by the equation

$$g_m(x(k))x(k+1) = f_m(x(k)) \qquad (1)$$

where $k \in \mathbb{N}$ is the discrete time variable, $x \in \mathbb{R}^n$ is the state vector, g_m is scalar polynomial function of degree not greater than $m \in \mathbb{N}$, and $f_m(x(k)) \in \mathbb{R}^n$ is a vector-valued polynomial function of degree not greater than m. We suppose that the initial condition $x(0)$ is not exactly known, but it is bound to belong to an ellipsoid $\mathcal{E}(P(0), c(0))$, i.e.

$$(x(0)-c(0))^\top P(0)^{-1}(x(0)-c(0)) \leqslant 1. \qquad (2)$$

The problem on which this article focuses is to find the smallest ellipsoid $\mathcal{E}(P(N), c(N))$ that, for all the valid initial conditions, bounds the state at a time $N > 0$, i.e. such as

$$(x(N)-c(N))^\top P(N)^{-1}(x(N)-c(N)) \leqslant 1. \qquad (3)$$

This problem can be decomposed into the iteration of an elementary problem on a single time step, i.e. finding the smallest $\mathcal{E}(P(k+1), c(k+1))$ for a given $x \in \mathcal{E}(P(k), c(k))$.

3. MAIN RESULT

As stated, g_m and f_m are polynomials in the state variable $x(k)$, with $x = [x_1,\, x_2,\, \dots x_n]^\top$ (dropping the dependency from the time variable in order to simplify the notation). As we are going to deal with polynomials up to degree m, we define also the vector $\chi \in \mathbb{R}^\rho$ which contains all the possible monomials obtainable from x from degree 0 up to m (for example, if $n = 2$, $m = 2$, then $\chi = [x_1,\, x_2,\, x_1^2,\, x_1x_2,\, x_2^2,\, 1]^\top$). We have that

$$\rho = \binom{n+m}{n}. \qquad (4)$$

In this way, any polynomial in the variables of x up to degree m can be formulated as a linear function of χ; so namely we have

$$f_m(x(k)) = F^\top \chi(k),\;\; g_m(x(k)) = G^\top \chi(k) \qquad (5)$$

with $F \in \mathbb{R}^{\rho\times n}$, $G \in \mathbb{R}^\rho$. Moreover, it is also possible to express polynomials up to degree $2m$ as a quadratic form with respect to χ, i.e. $p(x) = \chi^\top \mathcal{P} \chi$, with $\mathcal{P} = \mathcal{P}^\top \in \mathbb{R}^{\rho\times\rho}$. As reported in the literature related to sum of squares problems (Parrilo [2003]), this quadratic expression of a polynomial is not unique, due to the fact that different products of monomials can yield the same result, for example x_1^2 is either x_1^2 times 1 or x_1 times x_1. This implies that there exist linearly independent slack matrices $Q_k = Q_k^\top \in \mathbb{R}^{\rho\times\rho}$, with $k = 1, \dots, \iota$ such as $\chi^\top Q_k \chi = 0$. The number of such matrices is

$$\iota = \frac{1}{2}\left(\binom{m+n}{m}^2 + \binom{m+n}{m}\right) - \binom{n+2m}{2m}. \qquad (6)$$

This implies that, for a given $\mathcal{P}$, a polynomial of degree $2m$ or less can be expressed as

$$p(x) = \chi^\top \left(\mathcal{P} + \sum_{k=1}^{\iota} \psi_k Q_k\right)\chi \qquad (7)$$

for any $\psi \in \mathbb{R}^\iota$, $\psi = [\psi_1,\, \psi_2,\, \dots\, \psi_\iota]^\top$.

Before formulating our main result, we report two lemmas which will be useful for its proof.

Lemma 1 (S-procedure (Boyd et al. [1994])). *Consider the vector $z \in \mathbb{R}^r$, and matrices $X = X^\top \in \mathbb{R}^{r\times r}$, $Y_k = Y_k^\top \in \mathbb{R}^{r\times r}$ for $k = 1, \dots, \nu$. Let $\tau \in \mathbb{R}^\nu$, $\tau = [\tau_1,\, \tau_2,\, \dots\, \tau_\nu]^\top$. Then it holds that $X - \sum_{k=1}^{\nu} \tau_k Y_k \succeq 0$, $\tau_i \geqslant 0 \Rightarrow z^\top X z \geqslant 0$, $\forall z | z^T Y z \geqslant 0$, $k = 1, \dots, \nu$.*

Lemma 2 (Schur complement (Boyd et al. [1994])). *Consider three matrices $A = A^\top \succ 0$, B, $C = C^\top$ with compatible dimensions. Then*

$$\begin{bmatrix} A & B \\ B^\top & C \end{bmatrix} \succeq 0 \Leftrightarrow C - B^\top A^{-1} B \succeq 0. \qquad (8)$$

We are ready now to formulate the following theorem, which basically yields a practical solution for the problem in 2.2 in its step-by-step formulation.

Theorem 3. *Consider the dynamical system in (1). If $x(k)$ is such that $(x(k)-c(k))^\top P(k)^{-1}(x(k)-c(k)) \leqslant 1$ is satisfied, then $(x(k+1)-c(k+1))^\top P(k+1)^{-1}(x(k+1)-c(k+1)) \leqslant 1$ is true under (1) if the following two inequalities hold:*

$$\begin{bmatrix} GG^\top - \Theta + \sum_{k=1}^{\iota} \psi_k Q_k & F - Gc(k+1)^\top \\ * & P(k+1) \end{bmatrix} \succeq 0 \qquad (9)$$

$$\Omega + \sum_{k=1}^{\iota} \phi_k Q_k \succeq 0 \qquad (10)$$

for some values of

- $\psi \in \mathbb{R}^\iota$, $\psi = [\psi_1,\, \psi_2,\, \dots\, \psi_\iota]^\top$;
- $\phi \in \mathbb{R}^\iota$, $\phi = [\phi_1,\, \phi_2,\, \dots\, \phi_\iota]^\top$;
- $\Omega = \Omega^\top \in \mathbb{R}^{\rho\times\rho}$ *is such as* $\chi^\top \Omega \chi$ *is of degree* $2m-2$,

where

- $Q_k = Q_k^\top \in \mathbb{R}^{\rho\times\rho}$ *satisfies* $\chi^\top Q_k \chi = 0$ *for* $k = 1, \dots, \iota$;
- $\Theta = \Theta^\top \in \mathbb{R}^{\rho\times\rho}$ *is such as* $\chi^\top \Theta \chi = (\chi^\top \Omega \chi)\left(1 - (x-c(k))^\top P(k)^{-1}(x-c(k))\right)$.

Proof. We start by rewriting the expression $(x(k+1) - c(k+1))^\top P(k+1)^{-1}(x(k+1)-c(k+1)) \leqslant 1$; multiplying both sides by $g_m(x(k))^2 = \chi(k)^\top GG^\top \chi(k)$ and replacing $g_m(x(k))x(k+1)$ with $f_m(x(k))$ according to (1), we get $\chi(k)^\top GG^\top \chi(k) - (F^\top \chi(k) - c(k+1)G^\top \chi(k))^\top P(k+1)^{-1}(F^\top \chi(k) - c(k+1)G^\top \chi(k))^\top \geqslant 0$.

By using the Schur complement (Lemma 2), thanks to the fact that $P(k+1) \succ 0$, this is equivalent to

$$* \begin{bmatrix} GG^\top + \sum_{k=1}^{\iota} \psi_k Q_k & F - Gc(k+1)^\top \\ * & P(k+1) \end{bmatrix} \begin{bmatrix} \chi(k) & 0_{\rho\times n} \\ 0_{n\times 1} & I_n \end{bmatrix} \succeq 0. \qquad (11)$$

We would like to have this last expression verified (implying that $x(k+1)$ is inside the ellipsoid $\mathcal{E}(P(k+1), c(k+1))$) when $x(k)$ belongs to another ellipsoid, i.e. when $1-(x(k)-c(k))^\top P(k)^{-1}(x(k)-c(k)) \geqslant 0$. For this we can use the S-procedure (Lemma 1), in the case of $r = 1$, and we decide to employ an $x(k)$-varying multiplier $\tau(x) \geqslant 0$; namely we choose $\tau(x)$ as a polynomial of degree $2m-2$ in $x(k)$, i.e. $\tau(x) = \chi^\top \Omega \chi$, and in this way $\tau(x)(1 - (x(k)-c(k))^\top P(k)^{-1}(x(k)-c(k))$ is of degree $2m$ and can be expressed as $\chi^\top \Theta \chi$. The condition $\tau(x) = \chi^\top(\Omega +$