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On Existence of Separable Contraction Metrics for Monotone Nonlinear Systems ⋆

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Abstract: Finding separable certificates of stability is important for tractability of analysis methods for large-scale networked systems. In this paper we consider the question of when a nonlinear system which is contracting, i.e. all solutions are exponentially stable, can have that property verified by a separable metric. Making use of recent results in the theory of positive linear systems and separable Lyapunov functions, we prove several new results showing when this is possible, and discuss the application of to nonlinear distributed control design via convex optimization.

Keywords: Distributed control, Nonlinear systems, Convex optimization.

1. INTRODUCTION

Many emerging applications of system analysis and control involve large networks of interconnected nonlinear dynamic systems: smart-grid power systems, traffic management with autonomous vehicles, internet congestion control, and analysis of biological signalling networks, to name but a few. In order to make analysis methods scalable to large networks and robust to node dropouts or additions, it is crucial to be able to understand the overall network behaviour by way of conditions on just the local node dynamics and their interactions with immediate neighbours.

The traditional method for stability analysis makes use of a Lyapunov function: a positive-definite function of the system's state that decreases under flows of the system. Finding Lyapunov functions for arbitrary large-scale systems is generally intractable, but can be greatly simplified if one restricts the search to *separable* Lyapunov functions, in particular sum-separable, i.e. $V(x) = \sum_i V_i(x_i)$, or max-separable, i.e. $V(x) = \max_i V_i(x_i)$, where x_i denotes the state of the i^{th} subsystem, and x denotes the concatenation of all subsystem states (Dirr et al., 2015).

There has been substantial work recently on establishing when such separable Lyapunov functions should exist. In the case of linear positive systems, i.e. linear systems for which the non-negative orthant is flow-invariant, the Perron-Frobenius theory gives separable Lyapunov functions (Berman and Plemmons, 1994). More recently, several researchers have taken advantage of these properties to dramatically simplify problems of decentralized control design and system identification (Tanaka and Langbort, 2011), (Tanaka and Langbort, 2013), (Colombino et al., 2015), (Rantzer, 2015), (Umenberger and Manchester, 2016). In a recent paper, fundamental results about separable Lyapunov functions for positive systems have been extended to linear time-varying systems (Khong and Rantzer, 2016).

For nonlinear systems . Recent work has focused on establishing conditions on existence of separable Lyapunov functions, especially in connection with input-to-state stability properties Dashkovskiy et al. (2010), Ito et al. (2012) and monotone systems Dirr et al. (2015), the natural nonlinear generalization of a linear positive system Smith (1995)

Contraction analysis generalizes techniques from linear systems to nonlinear systems. Roughly speaking, a system is contracting if the linearization along *every* solution is exponentially stable Lohmiller and Slotine (1998). This property is verified by the existence of a *metric*, which can be taken to be of Riemannian form though others are possible. This idea can be traced back to Lewis (1949) and has been explored in more detail recently by Forni and Sepulchre (2014) and extended to analysis of limit cycles by Manchester and Slotine (2013). Advantages of contraction analysis include the fact that questions of stability are decoupled from knowledge of a particular solution, unlike Lyapunov theory, and because the object of study is a family of linear time-varying systems, many familiar results can be extended to nonlinear systems.

A natural question to ask is when these contraction conditions can be verified in a scalable way. It was noted in Coogan (2016) that some commonly used metrics based on l^1 and l^∞ norm are naturally separable, a property which was used implicitly in several papers on networked system analysis and control, e.g. (Russo et al., 2011; Como et al., 2015). However, this does not answer the question when such a separable metric exists for systems which are known to be contracting with respect to a (not necessarily separable) metric, as in the case of linear positive systems.

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Recently, the technique of control contraction metrics has been introduced, which extends contraction analysis to constructive control design (Manchester and Slotine, 2017). In fact, synthesis conditions can be transformed to a convex problem: a set of pointwise linear matrix inequalities, which can be verified using sum-of-squares programming (see e.g. Aylward et al. (2008)). Despite being convex, these conditions do not generally scale to very large systems.

It was recently shown by Stein Shiromoto and Manchester (2016) that if one restricts the search to sum-separable control contraction metrics, then the problem of distributed control synthesis can be made convex. This can be considered an extension of the results of Tanaka and Langbort (2011) to a class of nonlinear systems.

Another motivating application is nonlinear system identification with guaranteed model stability, building on the results of Tobenkin et al. (2017). Recent results allow scalable computation for linear systems (Umenberger and Manchester, 2016). With separable contraction metrics, these could be extended to identification of large-scale nonlinear systems.

2. NOTATION AND PRELIMINARIES

For symmetric matrices A, B the notation $A \geq B$ ($A > B$) means that $A - B$ is positive semidefinite (positive definite), whereas for vectors $x, y \in \mathbb{R}^n$, the notation $x \geq y$ denotes element-wise inequality. The non-negative reals are denoted $\mathbb{R}^+ := [0, \infty)$, and the natural numbers from 1 to n are denoted $\mathbb{N}_{1,n}$. A smooth matrix function $M(x,t)$ is called *uniformly bounded* if there exists $\alpha_2 \geq \alpha_1 > 0$ such that $\alpha_1 I \leq M(x,t) \leq \alpha_2 I$ for all x, t. Given a vector field $v : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ defined for $x \in \mathbb{R}^n, t \in \mathbb{R}^+$, we use the following notation for directional derivative of a matrix function: $\partial_v M := \sum_j \frac{\partial M}{\partial x_j} v_j$.

In this paper we consider time-varying nonlinear systems:

$$\dot{x} = f(x,t) \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector at time $t \in \mathbb{R}^+$, and f is a smooth function of x and at least piecewise-continuous in t, though these can be relaxed somewhat. Note that this system representation can include systems with external control or disturbance inputs $\dot{x}(t) = f(x(t), u(t), w(t))$, where for our purposes we absorb these into the time-variation in (8).

A dynamical system is *monotone* if for any pair of solutions x^a and x^b, $x^a(0) \leq x^b(0)$ implies $x^a(t) \leq x^b(t)$ for all $t \geq 0$, where $\leq$ denotes component-wise inequality. This property can be generalized to partial-orderings based on arbitrary cones, however the positive orthant is the natural cone for the purposes of studying separability. A differential characterization of monotonicity is that the off-diagonal elements of $\frac{\partial f}{\partial x}$ are non-negative – this is implied by the *Kamke-Müller conditions* (Smith, 1995).

Internally positive linear systems form an important subset of monotone systems. A continuous-time linear system

$$\dot{x} = A(t)x$$

is positive (and hence monotone) if $A_{ij}(t) \geq 0$ for all $i \neq j$ and for all t. In the case of time-invariant systems, the following result is well-known (see e.g. Berman and Plemmons (1994), Rantzer (2015)).

Theorem 1. If A is positive and Hurwitz, then there exists $p_i > 0, q_i > 0$, and $d_i > 0$, $i = 1, 2, ..., n$ such that the following functions

$$V_p(x) = \sum_i p_i |x_i|, \tag{2}$$

$$V_q(x) = \max_i \{q_i |x_i|\}, \tag{3}$$

$$V_d(x) = \sum_i d_i |x_i|^2, \tag{4}$$

are Lyapunov functions for the system $\dot{x} = Ax$. Moreover, one can take $d_i = p_i q_i$.

Note that $V_d(x)$ is a quadratic Lyapunov function $V_d(x) = x' D x$ for which D is diagonal and $D_{ii} = d_i$, and V_p is linear on the non-negative orthant.

A recent paper partially extends these results to linear time-varying (LTV) systems:

Theorem 2. (Khong and Rantzer (2016)). A linear positive system $\dot{x} = A(t)x$, with $A(t)$ piecewise continuous and uniformly bounded for all $t \in \mathbb{R}^+$, is exponentially stable if and only if there exists a Lyapunov function $V(x,t) = x' P(t) x$ with P diagonal such that $\eta |x|^2 \leq V(x,t) \leq \rho |x|^2$ and $\dot{V}(x,t) \leq -\nu |x|^2$ for all $t \geq 0$ and some $\eta, \rho, \nu > 0$.

We utilize the following standard results of Riemannian geometry, see, e.g., Do Carmo (1992) for details. A Riemannian metric is a smoothly-varying inner product $\langle \cdot, \cdot \rangle_x$ on the tangent space of a state manifold $\mathcal{X}$; this defines local notions of length, angle, and orthogonality. In this paper $\mathcal{X} = \mathbb{R}^n$ and the tangent space can also be identified with $\mathbb{R}^n$. We allow metrics to be smoothly time-varying, and use the following notation: $\langle \delta_1, \delta_2 \rangle_{x,t} = \delta_1' M(x,t) \delta_2$ and $\|\delta\|_{x,t} = \sqrt{\langle \delta, \delta \rangle_{x,t}}$. We call a metric *uniformly bounded* if $\exists \alpha_2 \geq \alpha_2 > 0$ such that $\alpha_1 I \leq M(x,t) \leq \alpha_2 I$ for all x, t. For a smooth curve $c : [0,1] \to \mathbb{R}^n$ we use the notation $c_s(s) := \frac{\partial c(s)}{\partial s}$, and define the Riemannian length and energy functionals as

$$L(c,t) := \int_0^1 \|c_s\|_{c,t} ds, \quad E(c,t) := \int_0^1 \|c_s\|_{c,t}^2 ds,$$

respectively, with integration interpreted as the summation of integrals for each smooth piece. Let Γ be the set of piecewise-smooth curves $[0,1] \to \mathbb{R}^n$, and for a pair of points $x, y \in \mathbb{R}^n$, let $\Gamma(x,y)$ be the subset of Γ connecting x to y, i.e. curves $c \in \Gamma(x,y)$ if $c \in \Gamma$, $c(0) = x$ and $c(1) = y$. A smooth curve $c(s)$ is *regular* if $\frac{\partial c}{\partial s} \neq 0$ for all $s \in [0,1]$. The Riemannian distance $d(x,y,t) := \inf_{c \in \Gamma(x,y)} L(c,t)$, and we define $E(x,y,t) := d(x,y,t)^2$. Under the conditions of the Hopf-Rinow theorem a smooth, regular minimum-length curve (a geodesic) γ exists connecting every such pair, and the energy and length satisfy the following inequalities: $E(x,y,t) = E(\gamma,t) = L(\gamma,t)^2 \leq L(c,t)^2 \leq E(c,t)$ where c is any curve joining x and y. For time-varying paths $c(t,s)$, we also write $c(t) := c(t, \cdot) : [0,1] \to \mathbb{R}^n$.

A nonlinear system (8) is called contracting if *all* solutions are exponentially stable. A central result of Lohmiller and Slotine (1998) is that if there exists a uniformly bounded metric $M(x,t)$ such that