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Nonlinear optimal control of vintage capital lifetime and irreversible investments

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Abstract

Vintage capital models such as Solow model, Ramsey model, two-factor model, and models with endogenous technological change are discussed to demonstrate a variety of applications of nonlinear integral models to mathematical economics and finance. For each model an optimal control problem is formulated and basic results about the existence and asymptotic behavior of a solution are provided.

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1. Introduction

Integral dynamic models with delay

$$\overline{x}(t) = \int_{-\infty}^{t} K(\tau, t)\overline{m}(\tau)\, d\tau, \tag{1.1}$$

have been extensively used in various applied areas, including mathematical economics and finance starting from 1960s [14,15]. In applications to economic growth, the $n \times l$ kernel matrix $K(\tau, t)$ corresponds to a specific efficiency (productivity) function of capital, $\overline{x}$ is an n-vector of products, and $\overline{m}$ is an l-vector of productive capital units of various types and

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resources. The structure of $K(\tau, t)$ reflects different situations in economics and finance: its dependence on t describes a non-embodied (autonomous) technological change (TC), the dependence on $t - \tau$ illustrates the economic depreciation and physical deterioration of the capital, and the dependence on τ represents the TC embodied in capital. The case $\partial K(\tau, t)/\partial\tau > 0$ corresponds to a technical progress when new capital is more efficient. In economics such models are known as vintage capital models (VCMs). They represent a new prospective mathematical tool for modeling technological innovation. It is a fast growing area of research. Its strong impact on mathematical finance is motivated by efficient description of fundamental finance characteristics such as cost of capital, risk of investment decisions, dynamics of finance investments, market uncertainty, etc. The validity of VCMs on real data is provided, i.e., in [3,5,16].

In control theory, an $n \times n$ control matrix $\Lambda(\tau, t) = \{\lambda_{ij}(\tau, t)\}$, $0 \leqslant \lambda_{ij} \leqslant 1$, of the *intensities of capital use* can be introduced to reflect the varying load of different types of capital vintages in production of various types of products. In many cases there is a feedback in the input–output system and some of the outputs $\overline{x}$ are used as new investments: $\overline{m} = Y\overline{x}$. The matrix $Y(\tau, t)$ is another control that represents the *distribution of product* in the constructed model. If $Y(\tau, t) = Y(\tau)$, then the investment decisions made at the time τ of the new capital creation are irreversible [13] and cannot be changed later. Then the model (1.1) becomes

$$\overline{x}(t) = \int_{-\infty}^{t} K(\tau, t)\Lambda(\tau, t)Y(\tau)\overline{x}(\tau)\,\mathrm{d}\tau. \tag{1.2}$$

In this case it is not allowed to change the type of machines that are already installed or change them to produce another product type. The equipment can be scraped only when it becomes obsolete. Assuming that the TC is progressive ($\partial K(\tau, t)/\partial\tau > 0$), a natural requirement is the use of the newest and most efficient capital. It leads to a special form of the two-dimensional control Λ:

$$\Lambda_{ij}(\tau, t) = \Lambda_{ii}(\tau, t) = \begin{cases} 1, & a_i(t) \leqslant \tau \leqslant t, \\ 0, & \tau < a_i(t), \end{cases} \tag{1.3}$$

which is now determined by a one-variable control a that describes a moment when a capital unit is removed from operation. Control form (1.3) is important in capital renovation. It decreases the dimension of the control problem (1.2), but creates an essential nonlinearity. Model (1.2) turns to be a model with unknown delay:

$$\overline{x}(t) = \int_{a(t)}^{t} K(\tau, t)Y(\tau, t)\overline{x}(\tau)\,\mathrm{d}\tau. \tag{1.4}$$

Model (1.4) can be classified as a model with a Leontieff-type production function (PF) (clay–clay). In economics such integral models describe VCMs that will be discussed in Section 2. If K depends on other economic functions such as capital, labor, etc., then we obtain putty–clay models with multi-factor PFs. Examples of such models will be given in Section 3.

Partial differential equations (PDEs) are another alternative mathematical tool used for modeling dynamic systems with delay. They have been used in [1] to study VCMs with